Beyond Cookies:

UNDERSTANDING VARIOUS DIVISION MODELS

Build on teachers' and students' understanding of division by emphasizing partitive and measurement models and strategies for writing quality division story problems.

By Cindy Jong and Robin Magruder

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James had twenty-four cookies to split evenly among a total of three people. How many cookies did each person get? is an example of a typical division story problem that teachers often create and students often solve. Although this story problem can provide a context for students to understand a division process, we believe teachers can further extend students' understanding by exposing them to a variety of contexts. Having a deeper understanding of division derived from multiple models is of great importance for teachers and students. For example, students will benefit from a greater understanding of division contexts as they study long division, fractions, and division of fractions.

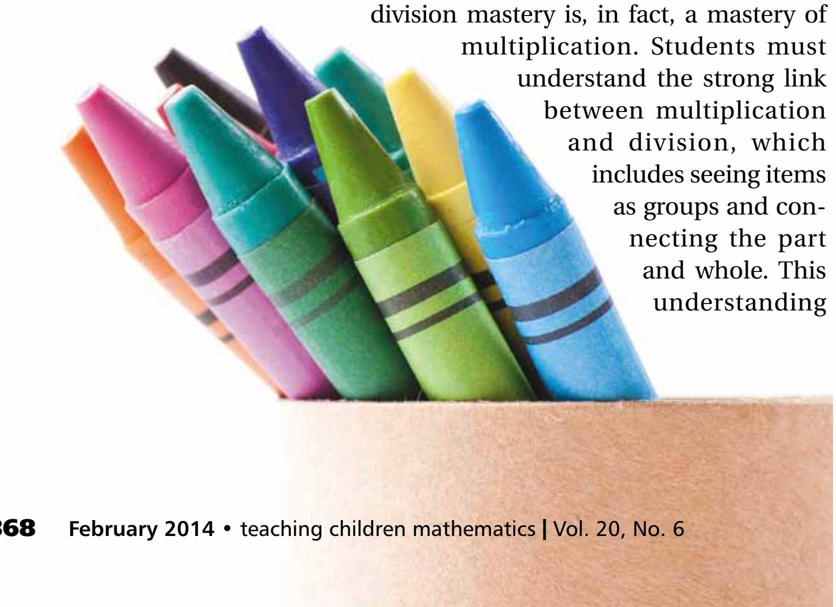
The purpose of this article is to build on teachers' and students' understanding of division, with an emphasis on the partitive and measurement division models. We begin by offering a rationale for examining division and then describe the two division models. Next, we discuss preservice teachers' understanding of the division models, and we present strategies for writing quality division story problems. We conclude by presenting engaging tasks that can be used with teachers to cultivate their understanding of the two division models.

DIVISION STORY PROBLEMS MUST EXPLICITLY STATE THAT THE SHARES ARE EQUAL.

Rationale: Why division?

An essential component of the elementary school mathematics curriculum that is formally introduced in third grade (CCSSI 2010; NCTM 2006), division is a basic operation closely associated with multiplication, and it is a skill required for understanding fractions and algebraic concepts. Whether students are sharing treats with siblings or splitting into groups for a game, they often encounter division scenarios outside of school. Because of these real-life experiences, it is natural for teachers and students to think of division within the context of food. For example, we have seen teachers use the children's book *The Doorbell Rang* (Hutchins 1989) to create a context for division that relates to student experiences. In this book, two children are eager to share their freshly baked cookies until they realize that as the size of the group increases, the number of cookies for each person decreases.

Teachers can emphasize a conceptual understanding by representing division in multiple ways and using division in different contexts (de Groot and Whalen 2006; Lamberg and Wiest 2012). Providing more contexts for division will enhance students' conceptual understanding of division by helping them think flexibly. The National Council of Teachers of Mathematics (NCTM 2000) suggested that as students learn multiplication and division, they must understand their relationship and multiple meanings. Van de Walle and Lovin (2006) described the close relationship between multiplication and division, suggesting that as students practice division problems, they may actually be using multiplication facts. We suggest that the key to division mastery is, in fact, a mastery of multiplication. Students must understand the strong link between multiplication and division, which includes seeing items as groups and connecting the part and whole. This understanding of the part and whole is an essential foundation for understanding fractions (Van de Walle and Lovin 2006). Familiarity with partitive and measurement division models helps students understand real-world scenarios, strengthens their understanding of the relationship between multiplication and division, and provides a useful background for the study of fractions (Van de Walle and Lovin 2006).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Division models: Partitive and measurement

> If a teacher wants to share a bag of fifteen cubes with five students equally, how many cubes will each student receive?

As the typical example above demonstrates, in partitive division, the whole (the dividend) and the number of groups (the divisor) are known, but the equal number of items in each group (the quotient) is missing. Partitive division problems are the most common model of division presented by teachers in the United States, and according to Lamberg and Wiest (2012), students are familiar with the context because from early childhood on, they have shared items with others.

In measurement division (also referred to as *quotative* in some texts), the whole and the equal number of items in each group are known, but the number of groups is unknown. Here is a typical measurement division problem:

> A teacher wants to share a box of fifteen crayons by giving each student three crayons. How many students will receive crayons?

In the measurement division model, each person receives the same quota of items, and repeated subtraction is the most common method for approaching these problems. Presenting examples of the measurement model of division is essential because the model is useful for understanding the division of fractions in the upper grades (Van de Walle, Karp, and Bay-Williams 2013). In future years, students may encounter such problems as this:

> A carpenter has a 7 1/2 foot board that she must cut into 3/4 foot sections to build picture frames. How many pieces will she be

able to cut, and how many picture frames will she be able to build?

This fraction problem is easiest understood by the measurement model, with students measuring out three-fourths-of-a-foot sections of the board.

Preservice teachers' understanding of division

In the spring of 2011, we collected instructional and assessment data in two mathematics methods courses. Fifty-five preservice elementary school mathematics teachers (PSTs) received two weeks of instruction examining the meaning of multiplication and division, creating story problems, and discussing strategies to teach multiplication and division. We noticed that few PSTs had been exposed to the measurement model of division and that it was a challenge for them to create story problems to represent the measurement model. Thus, we thought that sharing our experience and advice with (preservice) teachers and teacher educators on strategies for extending an understanding of division would be valuable to them.

When the preservice teachers completed an assessment on division (see **fig. 1**), approximately 30 percent of them had an incorrect or incomplete response. Several of the PSTs switched the meaning of the division models, created problems that did not match the representation, or had incomplete explanations. As an example of switched meanings, one PST wrote this:

> Sylvia has twenty-four cookies that she wants to separate equally among eight students. How many cookies does each student receive?

This example represents eight groups of three, rather than the indicated pictorial model of three groups of eight. We also observed three common trends in the assessment responses, which we outline below and for which we offer complementary suggestions to guide teachers to write quality story problems.

Story problem trends and suggestions

1. Questions must be clear.

Among preservice teacher questions in these courses, one unclear aspect was whether the person framing the question also received a share of the items being divided. When framing questions, we suggest making explicit whether the person dividing the items is also receiving a share. For example,

> Mitch has nine toy cars that he wants to share equally with two friends and himself. How many cars will each child receive?

Even better, have a distinct person do the dividing. For example, have a teacher divide groups of students, or have a coach divide players into small teams. In this type of story problem, the coach who is dividing the teams is clearly not going to be a player on the teams.

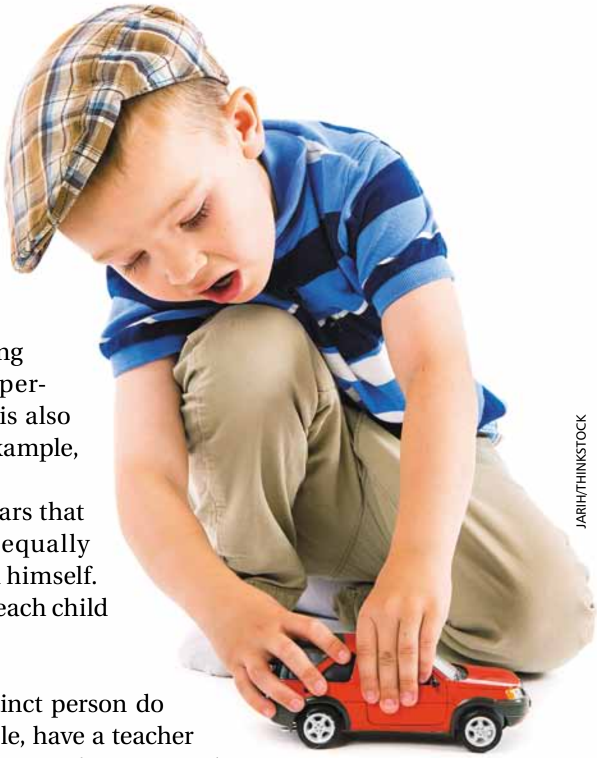

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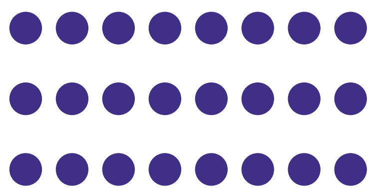
FIGURE 1

About 30 percent of the preservice teachers had an incorrect or incomplete response to the division assessment: They reversed the meaning of the models, created problems that did not match the representation, or gave partial explanations.

Two students, Kristin and Ray, were each using counters to help them solve simple story problems for division from their third-grade mathematics books. Both children started with a pile of **24 counters**. When they had finished, they each had the following on their desks:



However, the two children did not use the same procedure to arrange their counters this way. Each was doing a different problem in the text.

Kristin wrote 24 ÷ 3 = $\square$.
Write a story problem that Kristin might have been doing.
Which **model** of division did she use? (How do you know this?)

Ray wrote 24 ÷ 8 = $\square$.
Write a story problem that Ray might have been doing.
Which **model** of division did he use? (How do you know this?)

TABLE 1

The first two suggestions below are similar to observations that Polly and Ruble (2009) made when they examined division story problems created by third graders: The story problems were unclear, called for subtraction instead of division, and were inexplicit about equal groups. To avoid such errors with students, teachers must create quality story problems.

Suggestion	Partitive example	Measurement example
Clarity (Who receives a share?)	A school principal is taking 56 students on a field trip using 7 vehicles. How many students must be placed in each vehicle?	A school principal is taking 56 students to a museum that requires groups of 8 students in each tour group. How many groups from the school will be touring?
Equality of groups	A science teacher has 100 cL of water for an experiment, and she has 10 groups. If each group receives the same amount of water, how much will it receive?	A science teacher has 100 cL of water and wants to give each group 10 cL. How many groups will receive water for its experiment?
Variety of contexts	An artist has 64 oz. of clay and must make 4 sculptures. If each sculpture needs the same amount of clay, how much clay can be used for each?	A builder is using a 30 foot long piece of board as part of a deck. The board is cut into 6 foot sections. How many sections can be made from the board?

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2. Groups must be equal.

Many story problems that PSTs created (more than 50 percent) were ambiguous division situations, because the wording or questions did not require equal groups. The following was a typical question with this error:

> Mary has twenty-four dollars to share with her two brothers. How much does each child receive?

The error in this problem is an unclear emphasis on equality of shares. Mary could take twenty-four dollars, give one brother five dollars and the other ten dollars, and then keep the rest of the money for herself if equality is not emphasized. Additionally, several PSTs did not make it clear who was receiving a share. For example, PSTs wrote such problems as the following without stating that each child must receive the same amount of money:

> Mary, Bob, and Sue earned twelve dollars that they must share. How much did each child earn?

Story problems must explicitly state that the shares are equal. Emphasis on the equality aspect of division is a helpful backdrop to exploring fractions with children because of the equality of fractional parts.

3. Include a variety of contexts.

After spending class time discussing a variety of contexts, we were somewhat surprised to see such a high percentage (approximately 78 percent) of the preservice teachers include food in their story problems, and cookies in particular (55 percent). The majority of preservice teachers created problems that involved putting individual items into groups; these items included toys, pieces of fruit, money, school supplies, and students. Some preservice teachers created story problems to pique student interest, for example, beads to create a bracelet, guests at a birthday party, or horses in corrals. Only one preservice teacher created a problem that did not involve individual items but instead related to dividing a twenty-five–foot rope into five-foot–long segments (measurement). Alexander and Ambrose (2010) suggested discussing division in reference to items that are purchased or used as a set in a specific quantity, such as a dozen eggs, a deck of cards, or a box of crayons. Additionally, we suggest using items that are typically measured, for instance, lengths of wood or rope cut into even sections.

If this assessment is used in the future, we recommend that (preservice) teachers be required to use contexts for story problems that do not involve food, so that a variety of contexts are explored. If teachers do not describe a variety of contexts, students may relate division to only limited contexts and may not develop much flexibility in their thinking. In addition to creating a variety of problems, teachers should present students with opportunities to generate their own story problems.

To create story problems that support students' understanding of both partitive and measurement models of division, questions must be clear, equality of groups must be explicit, and questions must include a variety of contexts, including some relating to student interests (see **table 1**).

VALUELINE/THINKSTOCK

When developing division story problems, make the questions clear and the groups equal, and draw the subjects from a variety of contexts.

Tasks for teachers

Two tasks that were helpful for preservice teachers (and that can be used in professional development and elementary school classrooms) are (a) modeling story problems with concrete objects and (b) generating a list of division contexts. The preservice teachers explored two meanings of division using Unifix® cubes to show how students would directly model each problem with the two distinct strategies. For example, directly modeling 10 ÷ 2 with a partitive story problem context would call for a sharing strategy in which the ten Unifix cubes would be essentially dealt out (as playing cards are typically shared) one by one among the groups to figure out how many were in each group, whereas directly modeling 10 ÷ 2 with a measurement story problem context would call for chunking the Unifix cubes into groups of two (with repeated subtraction) to determine the total number of groups.

To further understand the two division models, the preservice teachers were asked to create a variety of division story problems to represent 27 ÷ 4 within their small groups and pay special attention to the context (see **fig. 2**). After each small group had created a list of story problems, every group took a turn sharing its problem while the rest of the groups decided whether the story problem fit the partitive or measurement model of division and provided a rationale for their selection. Next, the class generated a list of division scenarios that could relate to students.

A division expression with a remainder was purposefully selected to bring up the meaning of the remainder as context dependent. For example, when dividing ribbon, the remainder could be a fraction (i. e., 1/4 of a foot, or 3 inches) as

FIGURE 2

As a group writing task, preservice teachers were to create a variety of division story problems for a single expression with a remainder, purposefully selected to highlight remainders as context dependent.

Write as many story problems as you can that represent the following division expression. Keep the different meanings of division in mind, and use a variety of contexts (that will relate to students).

27 ÷ 4

→ reflect and discuss

"Beyond Cookies: Understanding Various Division Models"

Reflective teaching is a process of self-observation and self-evaluation. It means looking at your classroom practice, thinking about what you do and why you do it, and then evaluating whether it works. By collecting information about what goes on in our classrooms and then analyzing and evaluating this information, we identify and explore our own practices and underlying beliefs.

The following questions related to "Beyond Cookies: Understanding Various Division Models," by Cindy Jong and Robin Magruder, are suggested prompts to aid you in reflecting on the article and on how the authors' ideas might benefit your own classroom practice. You are encouraged to reflect on the article independently as well as discuss it with your colleagues.

- What division word problems do you typically ask your students? Do your word problems include both partitive and measurement models of division?
- In what ways will your students benefit from understanding both models of division?
- What division contexts might be of interest to your students and connect with their lives outside of school?

Tell us how you used Reflect and Discuss as part of your professional development. The Editorial Panel appreciates the interest and values the views of those who take the time to send us their comments. Letters may be submitted to Teaching Children Mathematics *at* ***tcm@nctm.org***. *Please include Readers Exchange in the subject line. Because of space limitations, letters and rejoinders from authors beyond the 250-word limit may be subject to abridgment. Letters are also edited for style and content.*

opposed to dividing money, with remainders as decimals ($0.25). In the case of dividing people into teams, splitting a human as a remainder makes no sense. Overall, experience with the two aforementioned tasks appeared to have enhanced the class discussions about the division models and the various contexts.

Teaching the teachers

American teachers often tell children that division is nothing more than repeated subtraction. This idea not only is insufficient but also hinders children's ability to construct an understanding of the part/whole relationships in multiplication and division (Fosnot and Dolk 2001, p. 53). Having a deeper understanding of division derived from multiple models is of great importance for teachers and students. This foundational understanding can help facilitate the learning of mathematics topics that tend to be a challenge for students, such as long division, fractions, and division of fractions (Hedges, Huinker, and Steinmeyer 2005; Van de Walle and Lovin 2006). Thus, we believe that teachers should learn more about the various models of division. Teachers' understanding of division models can be enhanced by engaging them in meaningful tasks, assessing their understanding, and providing clear guidelines for writing story problems. In particular, it is also critical that teachers expose students to a wide variety of contexts that are appropriate for division, reaching beyond food—especially beyond cookies—to help students make more connections and think flexibly. For students to develop a broader notion of division, they must be exposed to various contexts; this will happen only as teachers develop a broader context and multiple meanings of division.

Common Core Connections

3.OA.A.2
3.OA.A.3
SMP 4

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Cindy Jong, cindy.jong@uky.edu, is an assistant professor of math education at the University of Kentucky. She is interested in improving math teacher education to prepare elementary school teachers to teach math effectively to students. **Robin Magruder, rlmagruder@campbellsville.edu**, is an assistant professor of education at Campbellsville University. She has taught K–grade 12 math for more than a dozen years and is interested in examining the use of manipulatives in mathematics.