Maximizing Volume with Solids and Nets

Victoria L. Miles

One of the most common household polyhedra is a cereal box. For over 100 years, American companies like Kellogg's™ have packaged cereal in containers shaped like rectangular prisms. Why is a rectangular prism the most commonly used solid for holding cereal? Would another design be equally or more efficient? Students will explore these and other questions in this open-ended measurement activity.

To analyze how to maximize volume, students will investigate a household container of their choice. They will measure its dimensions, draw a two-dimensional net to represent the three-dimensional solid, and determine its surface area and volume. They will also design alternative solids with the same volume but less surface area. At the close of this curriculum-embedded performance-based assessment activity, students will summarize their findings in a report format of their choosing.

This activity qualifies as an authentic performance task because it presents students with a real-world problem that is set within a context of realistic constraints and possibilities. It also measures students' use of their knowledge and skills in new situations Wiggins and McTighe 2005).

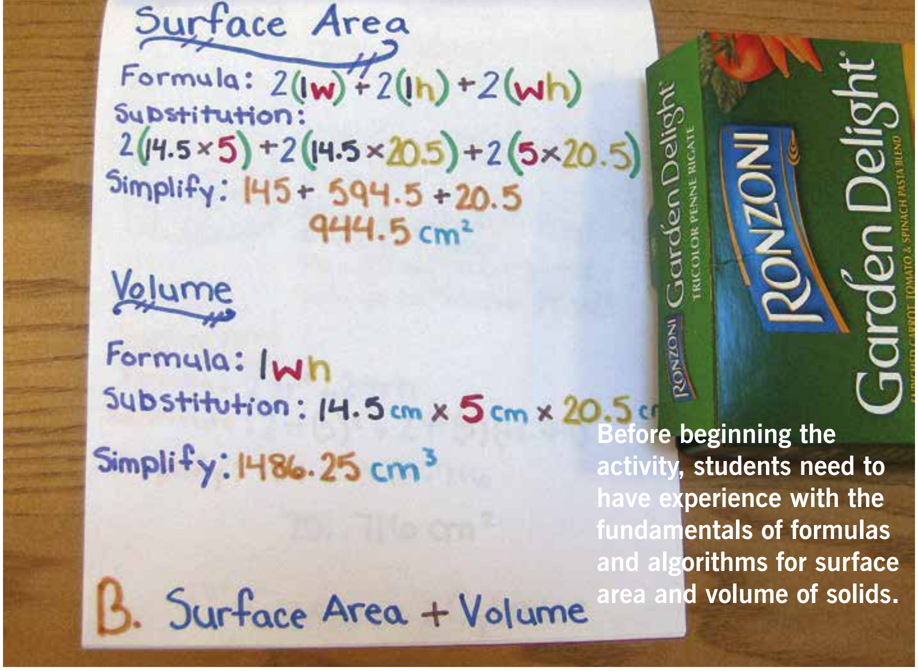
DAISY-DAISY/THINKSTOCK

According to the Common Core State Standards for Mathematics (CCSSM), in grade 5, students explore volume by filling rectangular prisms with unit cubes (5.MD.C.4). In grade 6, they solve problems involving prisms with integer and fractional dimensions (6.G.A.2) and represent solid figures made of rectangles and triangles using two-dimensional nets. They then use the nets to help them determine the surface area of these figures (6.G.A.4). In grade 7, students solve real-world and math problems involving circles (7.G.B.4). In grade 8, they use their knowledge of circles to solve problems involving such circular solids as cylinders and spheres (8.G.C.9). Grade 8 students learn to apply the Pythagorean theorem to solve problems involving cones and pyramids (8.G.B.7). In high school geometry, students move into using volume formulas for cylinders, pyramids, cones, and spheres to solve problems (HSG.GMD.A.3) (CCSSI 2010). Although I used this activity with my ninth-grade geometry classes, it could be modified for lower grades.

BACKGROUND LEARNING EXPERIENCES

This nonroutine project, a culminating activity in a unit on solid geometry, requires students to employ higher-order thinking skills. Using mathematical discourse and discovery learning throughout the unit will help

Edited by **S. Asli Özgün-Koca**, aokoca@wayne.edu, Wayne State University, Detroit. Readers are encouraged to submit manuscripts through **http://mtms.msubmit.net**.



Before beginning the activity, students need to have experience with the fundamentals of formulas and algorithms for surface area and volume of solids.

ensure that students gain an enduring understanding of the measurement concepts under study.

Assigning background tasks will benefit students before this project begins. One such activity is finding the nets for a cube. Eleven distinct nets, or two-dimensional patterns, exist that when folded will form a cube. Use large square graph paper or commercial manipulatives such as Polydrons™ to help students find cube nets. For students who learn best with technology, NCTM's Illuminations site offers a webpage at http://illuminations.nctm.org/Activity.aspx?id=3544 that will allow students to interact with the nets for a cube. Giving students time to wrestle with such hands-on activities as working with solids and nets will help them learn to reason abstractly and quantitatively, which is one of the Standards for Mathematical Practice (SMP 2).

Assigning other meaningful tasks before this activity commences can involve such sense-making processes as working with formulas and algorithms for surface area and volume of solids. These activities align with CCSSM, which states that students "know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems" (8.G.9; CCSSI 2010, p. 56). The best way for students to know mathematical formulas is not to memorize but rather to derive them. For example, to help students remember the formula for the surface area of a sphere, cut an orange in half and show students the largest cross section of the sphere, the interior of a "great circle." Outline four "great circles" using paper plates. Ask students to peel the orange into small pieces and cover the four great circles. Thus, the formula for the surface area for a sphere, $SA = 4\pi r^2$, will make sense to students.

SPECIFICS OF THE MAXIMIZING VOLUME PROJECT

Launch the project by showing students a container, such as a cereal box. Ask these questions:

- What is the mathematical name for this object?
- How many faces does it have?
- What shapes make up the faces?
- What is the difference between surface area and volume of this container?
- How would we measure the surface area of the container?
- What is the process for finding the volume?

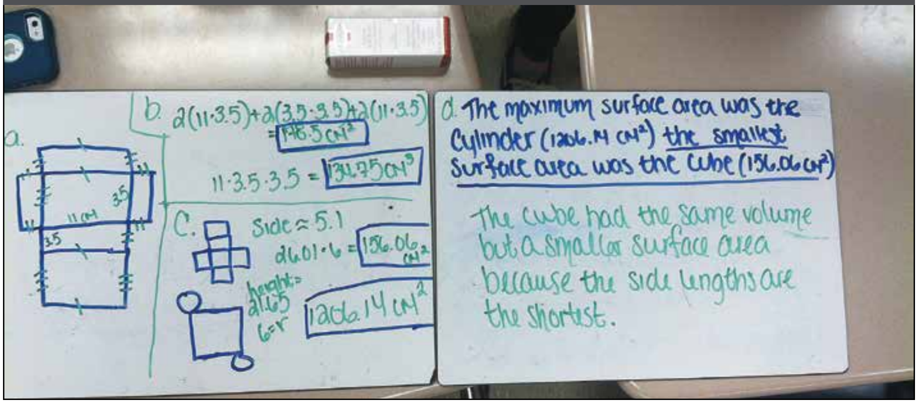
Ask students to envision the net, or the two-dimensional flat piece of cardboard, that when folded will result in the three-dimensional container you are holding. Invite students to sketch a possible net for the container. Use a pair of scissors and cut along the edges to reveal the net. Discuss any excess manufacturing material that the net may have. Cut off and discard extraneous pieces so that students can see the mathematical net versus the manufacturing net. Discuss how the net can be used to determine the surface area and volume of the container. Engage students in a discussion about surface area as the sum of the area of all the faces included in the net.

Ask students to conjecture why the product is packaged this way. Could another container hold the same amount but be shaped differently? Have students share their ideas of classifications of containers that would maximize the volume while minimizing surface area. Highlight the importance of this concept for manufacturing, keeping in mind profits and leftover material used in production.

Distribute the **activity sheet** and rubric and give students these instructions:

1. Complete a project to show what you know about surface area and volume.
2. Choose any container on which to focus.
3. Measure the dimensions of the object using convenient units.
4. Draw a mathematical net for the container and determine its surface area and volume.
5. Design other objects that have the same volume but require less surface area than the original container.
6. Conjecture why the other objects have the same volume but less surface area.
7. Create a product to showcase your findings.

Fig. 1 Assigning students a mock project allowed them to complete a trial run.



It will be time well spent to conduct a mock project in class before formally assigning the task so that students, working in groups, can experience a trial run (see **fig. 1** for a sample mock project). Distribute a container to each group, and have rulers, calculators, string, graph paper, chart paper, and markers available. Students will work together to follow the directions of the project. To conclude the mock project, students can participate in a gallery walk in which they view other groups' work and compare products and findings with their own. During the gallery walk, students will use the rubric to assess another group's work and provide feedback. I found that students were more motivated and better prepared to complete their own individual projects after having experienced the mock project.

DIFFICULTIES ENCOUNTERED

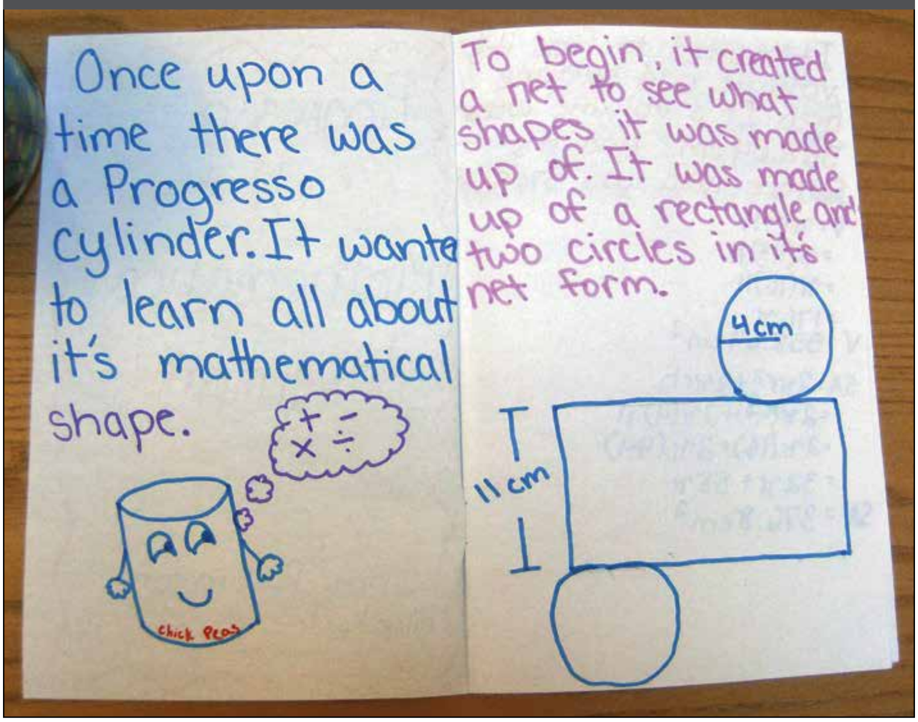
One area of confusion related to the net. For example, the manufacturing net of a cereal box might comprise eight rectangular pieces, as opposed to the mathematical net of a rectangular prism, which has six faces. It is important to alert students to exclude any extraneous pieces of manufacturing material from their net. This project assesses students' understanding of mathematical nets, which do not involve the overlap used in manufacturing.

A similar difficulty arose when students chose objects whose base or top was composed of two interconnected parts instead of one solid face. In this case, it is recommended that students tape the two linked pieces together to form one whole face.

Another problem that students had was remembering to keep the volume, and not the surface area, constant. For instance, a box with dimensions 10 in. × 5 in. × 12 in. would have a volume of 600 cubic inches and a surface area of 460 square inches. The goal for students is to design other containers that hold the same 600 cubic inches but have less surface area than 460 square inches. An error that several students made was keeping the 460 square inches constant and attempting to change the volume. It is important to reiterate to students that they need to keep the volume constant while varying the surface area.

Several students had difficulty in simultaneously finding two dimensions of an alternative container that would yield the same volume as the original container. For example, a student designing a cylinder to hold 600 cubic inches might get stuck on what radius and height combination to use. Suggest that students assign a convenient radius to the cylinder, so that the only unknown dimension is the height.

Fig. 2 This illustration was part of a student-authored treatise called *Chick Peas' Mathematical Adventure.*

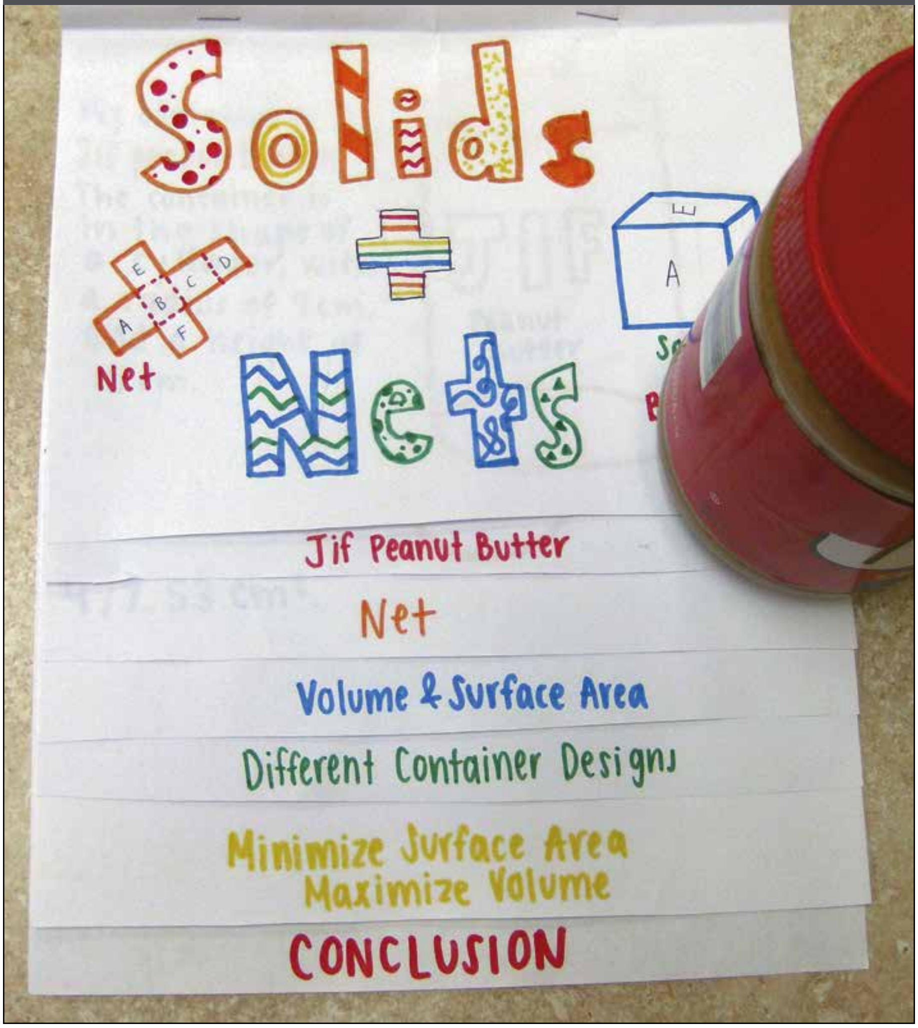


EXTENSIONS

- Give students some cardboard with which to construct the physical

container that will maximize volume and minimize surface area. They can then fill the original container with beans and dump them from the original container into the new container to show that it holds the same amount of contents.
- They can use a R.A.F.T. writing strategy (Dean 2006) to write a letter to the manufacturer of the product, suggesting alternative container shapes for the product. With R.A.F.T., students consider various perspectives while writing: role of the writer (Who are you? a consumer); audience (To whom are you writing? a company); format (In what format are you writing? a business letter); topic (What are you writing about? a design for a container that would maximize volume while minimizing surface area).
- Research the profession of manufacturing process engineering. Invite an engineer to come to class to discuss specifics of the work that he or she does. Students can ask prescripted questions of the engineer to verify the mathematics that manufacturing engineers use in their work.

Fig. 3 The food product of peanut butter found new life in a student's geometric flip-chart rendering.



RELEVANCE EQUALS ENGAGEMENT

I have assigned this project for two years and found that the second year's products were of higher quality and mathematical precision than the first year's. During the second year, 100 percent of my sixty geometry students turned in a project. I attribute the better results to the completion of the mock project.

The activity's exit prompt asked students to reflect on the value of completing a mock project in class. Student responses included these:

> I loved the mock project. It helped me greatly, because now I know exactly what to do at home.

> I liked walking around and seeing other people's work because it allowed me to learn new ways to solve problems.

> It helped me and now I am less likely to procrastinate because I know what I'm doing.

> I like the idea of the project. It lets me dig deeper because I am using the formulas to solve a real-life problem. I learn more in a project than filling in a bubble on a test.

When students perceive the relevance of an assignment, they will become more engaged. Research indicates that authentic tasks that involve choice and interest will yield more significant student learning outcomes. According to assessment expert Grant Wiggins (2014), authentic tasks offer realistic problems that target students' preferences and interests. Moreover, authentic tasks allow students to work collaboratively and assess essential understandings.

The solids and nets project embodies the elements of an authentic task. The project's components are known at the outset, and students work collaboratively and engage in rich discussion. Although it was more time-consuming to score sixty projects than run sixty forms through a scoring machine, it was enjoyable to assess them and provide feedback.

One student decided to revise and resubmit her project when she realized that she had some faulty understanding of the volume of her composite solid. Giving students time to use the rubric to assess another group's mock project paid off, resulting in higher-quality individual projects. One student created her project in the form of a storybook titled *Chick Peas' Mathematical Adventure* (see **fig. 2** on p. 249). Other product types included posters, accordions, PowerPoint® presentations, iMovies, and flip charts (see **fig. 3**). Students enjoyed sharing their favorite household products, and the containers ranged from decorative tea boxes to cylindrical mason jars. Providing project exemplars and suggesting formats sparked students' creativity.

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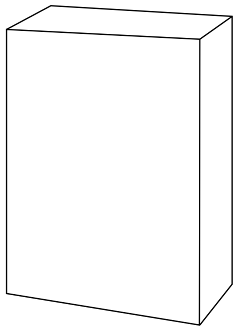
activity sheet

Name ______________________________

MAXIMIZING VOLUME WITH SOLIDS AND NETS



Your task is to investigate volume and surface area using a familiar household container. You might choose a rectangular prism, cylinder, or some other object that is used to hold cereal, soap, tea, coffee, and so on. You must design a more efficient container for the product, following the steps outlined below:

1. Measure the dimensions of the original container using convenient units.

2. Sketch a net for the container, which, if folded, could represent the object.

3. Determine the volume and surface area of your container.

4. Design three other containers, using different types of solids, to hold the same amount of product as your original container. The challenge is to design containers that hold the same volume but have less surface area than your original container.

5. Which solid maximizes volume while minimizing surface area? Why?

6. Summarize your findings in a report in a format of your preference. For example, make a Prezi©, a PowerPoint® presentation, a flip chart, a foldable booklet, a picture book, an iMovie, and so on. Be sure to meet the criteria of the project as detailed in the rubric.

Name ______________________________

RUBRIC FOR MAXIMIZING VOLUME WITH SOLIDS AND NETS

Criteria	Meets	Needs Improvement	Beginning
Dimensions	Appropriate, precise dimensions are identified.	Measurements lack precision. Units are not specified.	Dimensions are missing and/or are incorrectly measured.
Net	Net is drawn to scale; dimensions are labeled.	Net is lacking labels and/or components.	Net is incorrect or missing.
Surface area and volume of original container	Surface area and volume of original container are correct. Reasoning is shown. Units are correct.	Surface area or volume of original container is incorrect. Reasoning is shown.	Surface area and volume of original container are incorrect. No reasoning is shown, or the reasoning reveals a lack of understanding.
Object A	Surface area and volume are correct. Reasoning is shown. Units are correct.	Surface area or volume is incorrect. Reasoning is shown.	Both surface area and volume are incorrect. No reasoning is shown, or the reasoning reveals a lack of understanding.
Object B	Surface area and volume are correct. Reasoning is shown. Units are correct.	Surface area or volume is incorrect. Reasoning is shown.	Both surface area and volume are incorrect. No reasoning is shown, or the reasoning reveals a lack of understanding.
Object C	Surface area and volume are correct. Reasoning is shown. Units are correct.	Surface area or volume is incorrect. Reasoning is shown.	Both surface area and volume are incorrect. No reasoning is shown, or the reasoning reveals a lack of understanding.
Results/summary	Product that minimizes surface area while holding volume constant is correctly identified. Rationale is provided as to why this object holds the same amount but uses less packaging material. A correct net for this object is drawn.	Product that minimizes surface area while holding volume constant is correctly identified. Minimal rationale is provided. A mostly correct net for this product is drawn.	Product that minimizes surface area while holding volume constant is not correctly identified. No rationale is provided, or an incorrect net for this product is drawn.
Report (product)	Product is creative and high quality. Product demonstrates knowledge of concept.	Product may lack quality or creativity. Product demonstrates some knowledge of concept.	Product lacks quality and creativity. Product demonstrates a limited knowledge of concept.