Numeracy

Development and Intervention Guide

RENAISSANCE LEARNING™
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The *Numeracy Development and Intervention Guide* was written by Dr. Kenneth E. Vos, professor of education at St. Catherine University, St. Paul, Minnesota. With his strong ties to practitioners and classrooms, Professor Vos has crafted a guide that seamlessly blends research with practice. For more than 35 years, he has been preparing undergraduate and graduate students to teach mathematics, and in tandem working with classroom teachers. He has directed numerous programs and institutes for teachers on mathematics, best practices in the mathematics classroom, and assessment. The advice in this guide reflects the author's deep respect for the daily work of teachers. Dr. Kenneth E. Vos holds a Ph.D. in mathematics education from the University of Minnesota.

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by Dr. Kenneth E. Vos

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Contents

Overview

"Why do I have to learn these stupid multiplication facts?
I will never need to ever think of them quickly."

—third-grade student

The efficient recall of basic number facts is crucial to success in mathematics. Whether students can or cannot quickly and accurately remember number facts affects their attitudes about math. Students fall into two groups. Some know their number facts and are confident of their ability to succeed in mathematics. The others have trouble with number facts and declare their inability to ever succeed in mathematics. And we all know how that can be a self-fulfilling prophecy.

Ensuring that learners have their basic number facts down cold is easier said than done. An individual's learning style, ability to reason mathematically, and confidence in learning all contribute to rapid recall of basic number facts. A thoughtful plan of action based on reasonable expectations and practice will surely assist the learner to succeed. *The Numeracy Development & Intervention Guide* is designed to give a framework for such a plan.

Goals

The goals of this resource are twofold:

1. **To explore the broad contexts of acquiring basic number facts within the mathematics curriculum**
   Exploration of fundamental concepts of addition, subtraction, multiplication, and division is necessary before a learner can accomplish the recall of basic number facts. Understanding the nature of these operations, and their interrelationships, is key to success in this area. Each operation has unique features as well as subtle differences from the other operations that impact understanding and recall of the basic number facts for that operation.

2. **To develop strategies and techniques for intervention to enhance the rapid recall of basic number facts**
   Developing a toolkit of intervention strategies and techniques is paramount in giving learners confidence as they learn their basic number facts. While the four operations—addition, subtraction, multiplication, and division—are each unique, they are also interrelated; this fact allows us to use strategies and techniques across operations. This guide offers teachers a toolkit of ideas for each operation that you can reflect on, modify as needed, and implement in your classes.

Purposes

This resource was created with four specific purposes in mind.

1. **To investigate the relationship of number sense understanding and the acquisition of efficient recall of basic number facts**
   A solid foundation in number sense is necessary for efficient recall of basic number facts. This resource provides techniques and strategies for enhancing the understanding of number sense, which will then impact the recall of basic number facts. You'll find strategies and techniques for primary-age, intermediate-age, and middle/junior high-age learners. They are aligned with the development of the learner's ability to process and reflect on the mathematical concepts.

2. **To explain informal diagnostic strategies for detecting problems in recalling basic number facts**
   A brief plan of informal diagnostic strategies for specific situations is given in this resource. These are designed for a variety of contexts, including small-group settings, individual tutoring sessions, or at-home review with a parent or guardian at the dining room table. A diagnostic cycle is included to guide the process from beginning to completion.

3. **To offer constructed protocols that are ready for teacher or parent use, based on successful intervention techniques for basic number fact recall**
   The protocols are basic frameworks that include a specific objective, a realistic time period needed to complete the activity, a sequence of actions, a plan for modification for different students, and an evaluation strategy. These protocols are designed to be used by a classroom teacher for individual intervention or small group intervention; a parent or guardian within a home setting to supplement the intervention given by the classroom teacher; and by instructional aides as extra support for the classroom teacher's intervention.

4. **To present activities and games that can enhance the efficient recall of basic number facts**
   Classroom teachers have always supported the use of activities and games to motivate learners to practice their basic number facts. Such activities and games must, of course, reflect accurate mathematics, involve a balance of competition and cooperation, align with best practices in instruction—and be fun! The activities and games in this resource are all that, as well as being appropriate for all our target-age learners.

Structure of This Resource

The three sections of this book, *Early Numerical Skills (Grades 1–3)*, *Extending Numerical Skills (Grades 4–8),* and *Measures of Success*, present overlapping content. The first, *Early Numerical Skills*, focuses on primary-age learners' development of the mathematical understanding necessary for accurate recall of basic number facts. It begins with a brief overview of the foundation of numbers. Then it addresses the fundamental concepts necessary for the arithmetic operations of addition and subtraction, and offers a glimpse of those underlying multiplication and division as well. The section also includes informal diagnostic strategies for detecting when children are having problems; a brief list of strategies to enhance the efficient recall of addition and subtraction basic facts; and a set of twelve protocols that provide intervention activities. At the conclusion of this section, you'll find a useful one-page summary of advice from primary classroom teachers, called *Tips for Success*.

The second section, *Extending Numerical Skills*, focuses on intermediate- and middle/junior high-age learners. It gives a brief summary of the *Early Numerical Skills* section but includes a more advanced perspective on addition and subtraction. Next it moves on to provide a solid foundation for the understanding of the concepts of multiplication and division. The interrelationship of all four arithmetic operations is explored as it pertains to developing efficient recall of the basic number facts. At this level, as opposed to the primary level, the informal diagnostic techniques emphasize individual student interview settings. Aligned with these informal diagnostic strategies are possible alternative instructional techniques for recalling all four kinds of arithmetic basic facts, but especially multiplication and division basic facts. Fifteen protocols provide intervention for multiplication and division basic facts. This section also concludes with a handy *Tips for Success* sheet based on advice from intermediate and middle/junior high school math teachers.

The final section, *Measures of Success*, explains possible measures of success based on recall rate and accuracy for subgroups of the arithmetic operations. Here you'll find a summary chart of the possible benchmarks for addition, subtraction, multiplication, and division for learners of different ages.

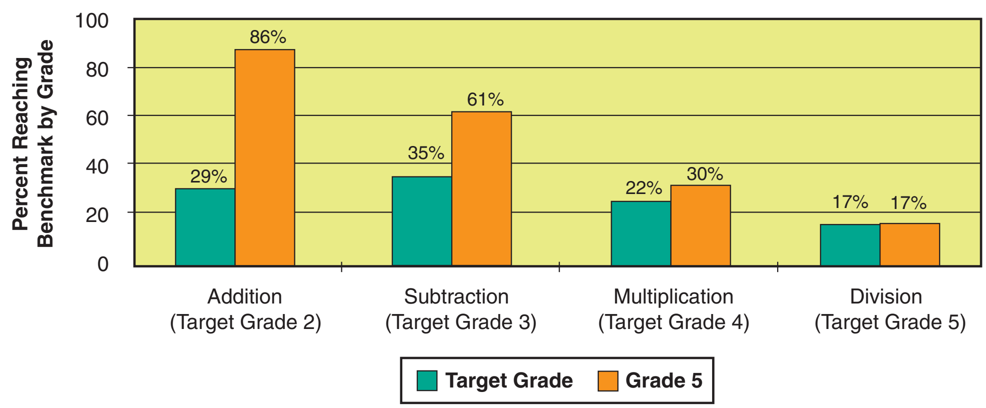
Basic Number Fluency Research

The impetus for this guide lies in the research on the importance of basic number fact fluency with respect to math achievement and on how students acquire automaticity. Students' preparation in the core number concepts and skills, while enabling them to handle basic requirements at their grade level, apparently has not been strong enough to provide an adequate foundation as they advance to tackle more challenging concepts and operations.

For the expanded version of the research section, see Math Facts Automaticity: *The Missing Element in Improving Math Achievement*, a white paper from Renaissance Learning, Inc.

Consider these statistics on math-skill mastery, derived from Renaissance Learning's online database of student mathematics activity in 2007–08 (nearly 200,000 students in all, nationwide):

Math Facts Mastery By Grade Level—End-of-Year Progress



The data show that not only do most students score below benchmarks in the target grades for the four operations, a distressingly high proportion do not even master them by middle school. On entering sixth grade, *fewer than a third have mastered multiplication facts, and fewer than a fifth have mastered division.* And since there is little if any focus on these facts after fifth grade, it is a safe assumption that many never master them at all (Baroody, 1985; Isaacs & Carroll, 1999).

This is not to say that most students do not know how to add, subtract, multiply, and divide. Clearly they do, or they would not score even as well as they do on benchmark assessments. But they have not achieved mastery—or more strictly speaking, they have not achieved automaticity, the essential foundation of computational fluency.

> As a student learns a new skill, he/she will become increasingly fluent… until it becomes automatic. *Automaticity* refers to the phenomenon that a skill can be performed with minimal awareness of its use (Hartnedy, Mozzoni, & Fahoum, 2005; Howell & Larson-Howell, 1990)….The ability … to *automatically* respond … may free limited cognitive resources that

> can be applied to the more complex computations and concepts.… If each component of a complex, multistep problem requires sustained attention, the completion of the problem will likely be impossible due to the limited capacity of working memory. (Axtell, 2009, p. 527, emphases added)

There seem to be two reasons why math facts mastery is so low:

- **The curriculum gap.** According to the National Mathematics Advisory Panel, "Few curricula in the United States provide sufficient practice to ensure fast and efficient solving of basic fact combinations and execution of standard algorithms" (National Mathematics Advisory Panel, 2008, p. 27). Therefore, most schools provide supplemental means of practicing math facts: flash cards, worksheets, and computer software.
- **Too little emphasis.** When math fact practice is provided, the practice may not be frequent enough, or systematic, ensuring students acquire conceptual understanding of numeracy concepts, practice skills with confidence, and then practice for automaticity.

Even though schools may believe they have math fact mastery "covered," true mastery will not occur without sufficient practice with automaticity as the goal.

Research suggests that more time must be devoted to deliberate, measurable practice in order to attain automaticity. There is a growing consensus that automatic recall of math facts—addition, subtraction, multiplication, and division—is an indispensable element in building computational fluency, and preparing students for math success, both present and future (Battista, 1999). Just as phonemic awareness and decoding are the crucial elements in learning to read, automaticity and conceptual understanding go hand in hand in mathematics development (Gersten & Chard, 1999). Failure to develop automatic retrieval, on the other hand, leads to mathematical difficulties (Bryant, Bryant, Gersten, Scammaca, & Chavez, 2008).

It is not enough that students simply "learn" their number facts—they must be committed to memory, just as letter sounds must be memorized in development of phonic automaticity (Willingham, 2009). Automaticity, also known as fluency, is a different kind of knowledge; it is "based on memory retrieval, whereas nonautomatic performance is based on an algorithm" (Logan, 1988, p. 494). Learning starts with understanding of concepts, to be sure, but memory skills must develop simultaneously. "Children need both procedural knowledge about how to do things and declarative knowledge of facts" (Pellegrino & Goldman, 1987, p. 31). Declarative memory, which recalls that things are so, not only speeds up the basic arithmetic operations themselves (Garnette & Fleischner, 1983), it also acts to "free up working memory capacity that then becomes available to address more difficult mathematical tasks" (Pegg, Graham, & Bellert, 2005, p. 50; Gersten, Jordan, & Flojo, 2005).

The key to automaticity is practice, and lots of it (Willingham, 2009). As the National Mathematics Advisory Panel Report (2008) points out,

> To prepare students for algebra, the curriculum must simultaneously develop conceptual understanding, *computational fluency*, and problem-solving skills.… Computational fluency with whole number operations is dependent on sufficient and appropriate *practice* to develop *automatic recall* of addition and related subtraction facts, and of multiplication and related division facts. (p. xix, emphases added)

To move a fact, or skill, from short-term to long-term memory requires "overlearning"—not just getting an item right, but getting it right repeatedly (Willingham, 2004). And retaining the memory for a long interval requires spacing out additional practice after initial mastery—emphasizing the importance of regular review of learned material (Rohrer, Taylor, Pashler, Wixted, & Capeda, 2005). Brain research indicates that repetitions actually produce changes in the brain, thickening the neurons' myelin sheath and creating more "bandwidth" for faster retrieval (Hill & Schneider, 2006).

For all of these reasons, the federal What Works Clearinghouse (WWC) recommends math facts fluency remediation for students *at all grade levels* in Response to Intervention (RTI) schools. The WWC math Practice Guide (Gersten et al, 2009) states:

> Quick retrieval of basic arithmetic facts is critical for success in mathematics. Yet research has found that many students with difficulties in mathematics are not fluent in such facts.… We recommend that about 10 minutes be devoted to building this proficiency during each intervention session. (p. 37)

This guide proves support for an optimal, systematic approach to helping students acquire math facts, and keep them, from the elementary grades through the middle grades.

Early Numerical Skills *(Grades 1–3)*

1

Early Numerical Skills

(Grades 1–3)

Introduction

Constructing the foundation for acquiring basic number facts in the primary grades requires a carefully designed sequence of developmentally appropriate strategies. A misunderstanding of a concept in these early years can cause difficulty for the learner later on. Therefore, it is key that learners have a solid grasp of the fundamental concepts of numerical thinking before they launch into the recall of basic number facts. This section gives a brief overview of these essential pre-number and number concepts.

Teaching the fundamental concepts of addition and subtraction consumes the vast majority of the curriculum time in the early primary grades. A blend of concept development and skill development with addition and subtraction is necessary for success in recalling basic number facts. Since the four arithmetic operations—addition, subtraction, multiplication, and division—are interrelated, it is a natural sequence to introduce multiplication and division concepts as well, soon after addition and subtraction concepts. In the later primary grades the operations of multiplication and division are introduced.

Despite this emphasis on concept development, some students do have trouble acquiring the basic number facts of addition and subtraction. Section 1 explores some informal strategies for diagnosing primary-age learners' misunderstandings with addition and subtraction, and presents a simple pedagogic cycle of error detection, diagnostic interview, prescription plan, assessment, and evaluation to aid teachers.

Section 1 also includes a brief discussion of strategies for enhancing the efficient recall of addition and subtraction basic facts. These strategies are not exhaustive or definitive but are sample activities selected by primary grade classroom teachers as having been successful for some of their learners.

In addition, twelve protocols appropriate for primary learners are included in this section. One protocol addresses counting, six focus on addition, and another five focus on subtraction. These protocols are not complete lesson plans or curriculum units, but should be considered templates with a specific focus that can be modified as needed. While designed primarily for small- and whole-group instruction, the protocols can also be used for individual tutoring by an educational aide or parent. Finally, at the end of the section you will find a list of ten Tips for Success. This list is a quick reference of best practices in teaching basic number facts to primary-age learners.

Pre-number Concepts, Skills, and Activities

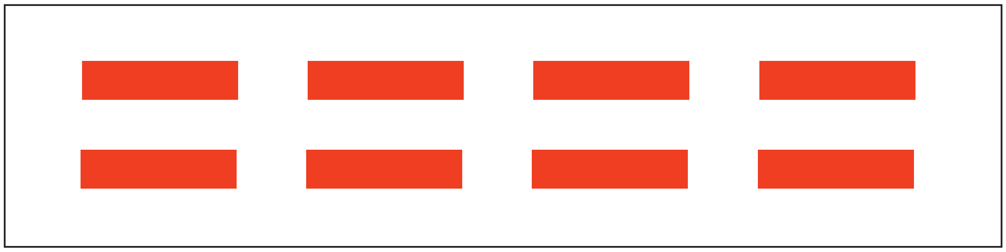
The process of understanding basic number facts begins early in the mathematics curriculum. It starts with a firm knowledge of key pre-number ideas and moves to learning numerals, which then allows the learner to develop these same key pre-number ideas more fully with numbers. There are at least four key pre-number ideas: we will focus here on order, size, one-to-one correspondence, and classification, or grouping. When exploring order without numbers, the early age learner decides what is *before* or *after* some other event or object. This understanding is necessary later for counting on and counting back to derive addition and subtraction basic number facts. Within the context of pre-number, the idea of size (more or less, bigger or smaller) is illustrated by viewing two groups of objects and determining which group contains more objects. This understanding is necessary later for learners to combine basic number facts, starting with a larger group and adding or subtracting the smaller group. The concept of one-to-one correspondence is really a matching concept embedded in a pre-number context. This understanding is necessary later when verifying specific subtraction number facts. In other words, five objects matched with four objects leaves one object remaining. The concept of classification involves sorting or grouping objects that could later be described. This understanding is necessary later when preparing two groups that will be combined into one (addition) or one group that will be separated into two (subtraction).

From these pre-number concepts flow the aligned number concepts. When the concept of number is introduced, these fundamental concepts of order, size, one-to-one correspondence, and classification are explored with number. So, order has the power of first, second, third, and so forth; in other words, the fundamental concept of ordinal numbers. Size takes on a deeper meaning, since now when viewing two groups of objects the decision of which group contains more objects yields a definitive value. One-to-one correspondence with numbers gives more opportunity to explore matching and comparison features associated with learning basic number facts. Classification creates the opportunity to combine similar groups (addition) or take a group and break it into two (subtraction). With these concepts established, the stage is set for development of the arithmetic operations.

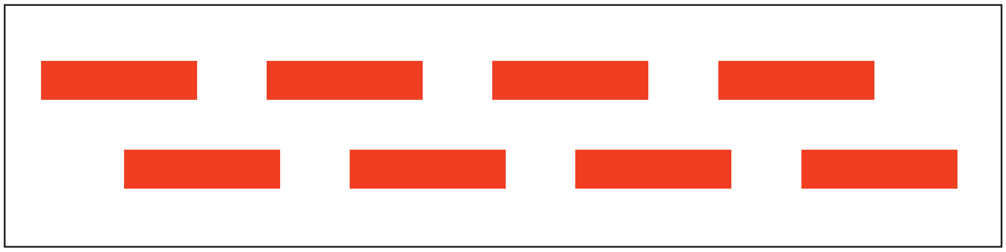
A basic principle of mathematics concept development is **touch it, see it, think it**. That is, students learn these concepts best when they follow the simple sequence of first using hands-on materials, then engaging with the concept in pictorial mode, and finally working with the concept in the abstract mode.

Touch it.	See it.	Think it.
concrete	**pictorial**	**abstract**

The best way for learners to first approach any new mathematical concept is with hands-on materials, or manipulatives. All the activities included here for developing the building blocks of addition and subtraction begin with manipulatives. However, as with any instructional technique, hands-on materials are not foolproof; they can lead to misunderstanding. For example, imagine a young child is presented with the following situation:



The child is asked, “Which row has more objects, or is there the same number of objects in each row?” Most children will answer that there are the same number of objects in the rows. However, when the one row is spread out as in the following situation, the children may encounter difficulty.



When asked, “Which row has more objects, or is there the same number of objects in each row?” many young children will say the bottom row has more objects. These children are not ready developmentally for comparison activities or matching situations. One of the protocols in this section (Sheep in the Pasture) will suggest activities that reflect the mathematical development of children who make this error.

The most challenging and subtle concept for young children is counting. It seems so simple. Young children enjoy counting their age, their toys, and their siblings. However, for accurate mathematical understanding they must “get” the concept of counting at a deeper level. Unfortunately, students sometimes learn the number sequence in the same way they learn the alphabet letter sequence—though rote memorization. This can lead to a basic misunderstanding of counting. The alphabet is not an ordered set of values, but an arbitrarily agreed upon sequence. The alphabet needs to be memorized. The number system, by contrast, is ordered. That is, if you know the value eight and its name but not the name of the next value, you still know its value is exactly one more than eight. Therefore, while memorizing the counting numbers is valuable, it can open the way to misunderstanding. For example, watch a young child count out five objects. Most likely the child will take an object, say “one,” and then put the object down; take the next object, say "two," and place that down. The child will continue to say the number and place the object on the surface until all five objects are on the surface. However, in this instance the child is not counting, but rather giving the order of the objects. The child says "two" but has only one object in hand. The child says "three" but only has one object in hand. The child is really indicating first, second, third, fourth, and fifth object.

Always check if a young child knows how to accurately count up, count down, and count on before introducing the concepts of addition and subtraction.

An accurate technique for teaching counting would be to have the child transfer the objects from one hand to the other, saying the value after each object has been transferred. In this case, when the child says "two," two objects are in the other hand. The child sees two, and says two. This technique is simple to teach and easy for young children to do.

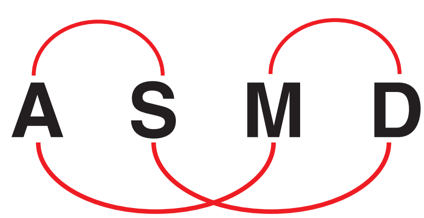
Counting is the keystone to success in learning addition and subtraction number facts accurately. Always check if a young child knows how to accurately count up, count down, and count on before introducing the concepts of addition and subtraction.

All four arithmetic operations are rooted in understanding how to count accurately and quickly. Successful counting also is the pathway to an in-depth understanding of number sense. Number sense gives power and confidence to the learner.

Number sense gives power and confidence to the learner.

Fundamental Concepts of Addition and Subtraction

The four arithmetic operations, addition, subtraction, multiplication, and division, are interrelated. Addition and subtraction are inverses of each other: for example, $2 + 3 = 5$ and $5 - 3 = 2$. Multiplication and division are inverses of each other as well: $2 \times 3 = 6$ and $6 \div 3 = 2$. Also multiplication can be viewed as repeated addition: $2 \times 31 = 62$ or $31 + 31 = 62$. Division can be viewed as repeated subtraction: $21 \div 7 = 3$; or $21 - 7$, $14 - 7$, $7 - 7 = 0$; or 21 minus 3 groups of 7 with no remaining values to substract.



These interrelationships of the four arithmetic operations are a key feature in understanding number sense, which assists in recalling basic number facts. Each arithmetic operation can be found from any other arithmetic operation. They are all aligned with each other. Primary-age learners should be comfortable moving among the four arithmetic operations.

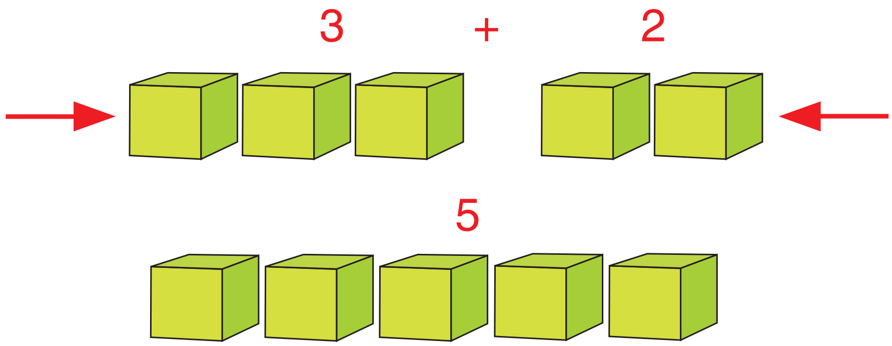
Addition

The concept of addition can be described as a joining together of two sets of objects. Addition is a binary operation that combines two values to form a result. The concept of addition allows us to join two groups to form a third group, the result. Primary-age learners usually obtain the number value of the third group by counting. Therefore, a beginning primary-age learner needs to be able to count on from a given value, count back, and count on after a pause. One of the protocols at the end of this section focuses on using counting to establish the result of joining two groups into a third group.

The concept of addition is developed in primary-age learners beginning with hands-on materials followed by pictorial settings and finally using abstract modes. The following activities involve hands-on materials appropriate for primary-age learners. Developing a complete understanding of the concept of addition requires a multitude of activities. These are just a few that may be helpful.

Counters

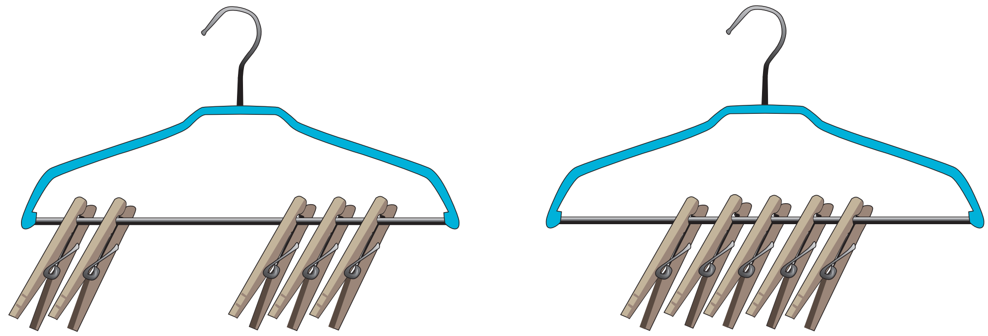
For young primary-age learners, teachers can construct situations using any group of objects to show the basic number facts up to 9. For example, to construct the addition of 3 and 2, a group of three counters (or objects) and a group of two counters (or objects) are placed apart from each other. The child would move the two groups together to form a single group. At this point, the child should count the total number of objects.



Later the child could draw the two groups and draw the combined group. The child could count the total number in the combined group. This would be the pictorial mode. Much later the child could write this situation as an entry to establish the recall of a basic number fact, $3 + 2 = 5$. Using counters is appropriate for all the basic addition number facts up to and including the value of 18.

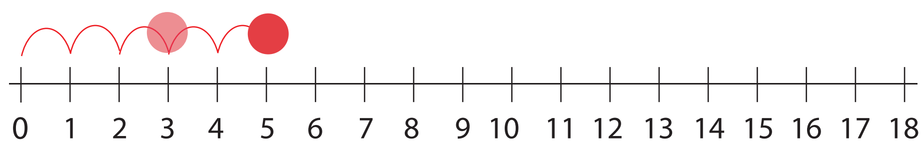
Computer Hanger

A computer hanger is a wire clothes hanger with a set of clip clothespins attached. Using a group of two clothespins on one end of the hanger and a group of three clothespins on the other end, the child would slide the two groups to the center of the hanger (showing the joining together) and count the result. Again, this simple device can be used to demonstrate all basic addition facts.



Number Line

A simple but powerful technique for teaching the concept of addition is a number line—a line with equally spaced numbers, starting at 1 (or zero) and progressing to 18.



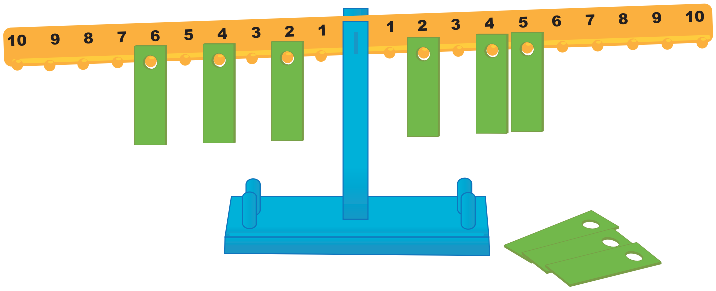
Using 3 + 2 as the basic addition fact, the child starts at zero and moves a marker three marks [landing on 3], pauses, and then counts two more marks to land on 5 as the result. This number line is appropriate for all basic addition facts with a result equal to or less than 18.

Number Pieces

Number pieces are constructed to clearly demonstrate the joining of two groups into a third group. The actual number pieces and a description of how to use them are found in the addition protocol "Number Pieces." *See Appendix for a template to create them.*

Math Balance

A math balance is a device that uses a lever and a fulcrum to demonstrate a balance of values. This math balance will be demonstrated in protocols for addition and subtraction (Addition Balance and Balancing Your Addends).

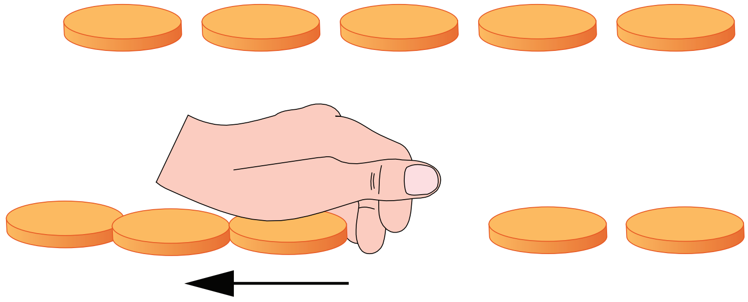


Subtraction

The subtraction concept seems to be more difficult for some primary-age learners to grasp. The notion of finding a result that is less than what is given can lead to confusion. Also, some methods of embodying the subtraction concept can have inherent flaws or limitations. One of the more common methods is called removal. Since the concept of subtraction entails breaking apart one group into two groups (the inverse of addition), the removal method illustrates this breaking apart action.

Counters

For example, 5 – 3 can be shown with counters as



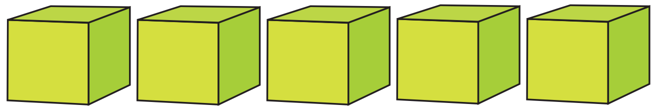
One of the difficulties with the removal method is that it is not easy to reconstruct the original situation to verify the result, since 3 counters were removed. The original situation could be any combination that results in 2 counters remaining, for example, 6 – 4.

Math Balance

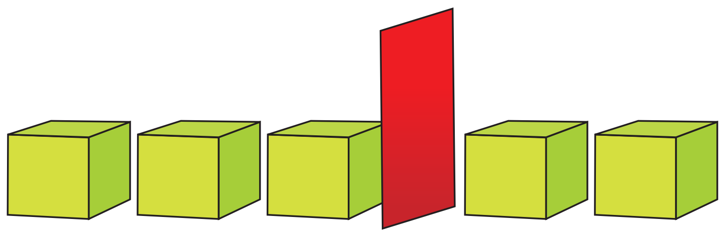
Another method that demonstrates the subtraction concept is called the missing addend. For example, 5 – 3 could be shown as 5 = 3 + [ ] , with students being asked to find the missing value, or addend. This clearly shows the inverse relationship of subtraction and addition. However, many primary-age learners struggle with this situation. The math balance is a good tactile device to demonstrate what happens in a missing addend situation. One of the subtraction protocols will develop techniques to demonstrate the missing addend basic facts.

Partition

A third method that is sometimes used to demonstrate the concept of subtraction is called the partition technique. A partition separates a set of objects into parts. For subtraction, the partition will separate the set of objects into two groups. For example, a learner is asked: 5 – 3 = [ ] and is given the set below.

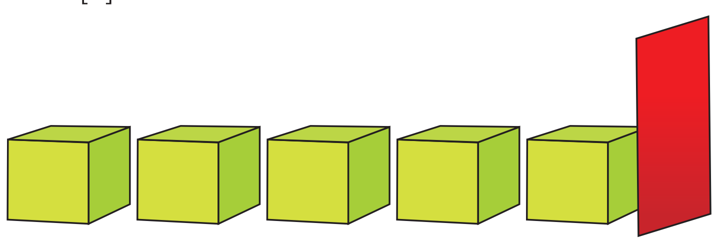


Using the objects and an index card, or a special partition card made by the child, the group of five objects can be partitioned as shown.

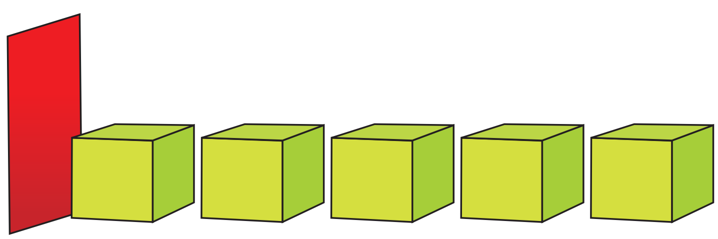


The left side shows three objects and the right side shows two objects (the result) while the partition vertical line could illustrate the equality symbol.

During the development of the subtraction concept, the zero value should be explored. For example, using the partition technique the child could show the result of 5 – 5 = [ ].



The child could also show the result of 5 – 0 = [ ].



All the activities described in the development of the addition concept can be modified to demonstrate the subtraction concept. Interchanging activities across addition and subtraction should enhance learners' understanding of the inverse relationship of these two operations. With this strong foundational understanding of addition and subtraction concepts, the child is now ready for the multiplication and division concepts.

Fundamental Concepts of Beginning Multiplication and Division

Multiplication and division concepts are not usually presented to early primary learners until some time has elapsed after they have developed their addition and subtraction concepts. Nonetheless, multiplication and division concepts are clearly related to the addition and subtraction concepts. This interrelationship gives another opportunity to emphasize the acquisition of number sense in another context. This section addresses just the beginning aspects of multiplication and division basic number facts. Section 2, *Extending Numerical Skills*, includes an expanded discussion of multiplication and division, with appropriate activities and strategies for acquiring these basic number facts.

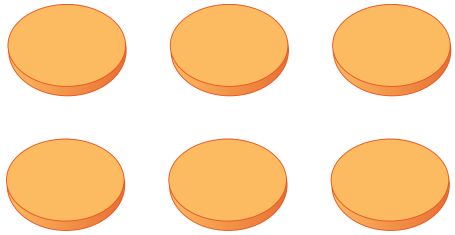
Multiplication

The concept of multiplication builds from the addition construct. The word multiplication is formed from two Latin words, *multus* and *plica*, loosely translated as "many fold," which refers to multiplication as repeated addition. Therefore, the concept of multiplication is usually demonstrated using addition number facts.

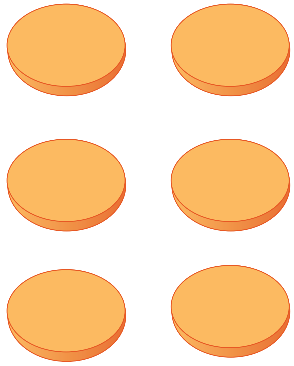
For example, 2 x 3 could be represented by two groups of three objects, or 3 + 3, that results in 6. However, it is also possible to read the expression 2 x 3 as two groups taken three times (2 + 2 + 2). Fortunately, multiplication is commutative and either interpretation is valid. This discussion will use the convention of 2 x 3 as two groups of three objects, or 3 + 3.

Array

Another way to demonstrate the concept of multiplication is to display an array of entries composed of rows and columns where the number of rows is the first factor of the multiplication fact and the number of columns is the second factor. For example, 2 x 3 would be shown in this way:



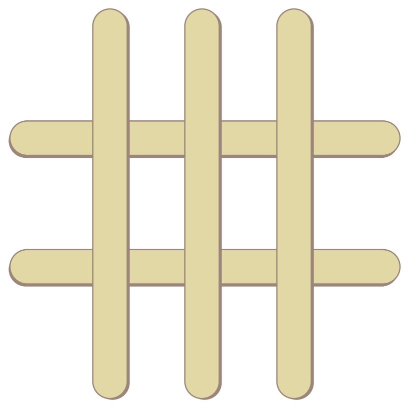
This diagram can be rotated 90° to show 3 x 2.



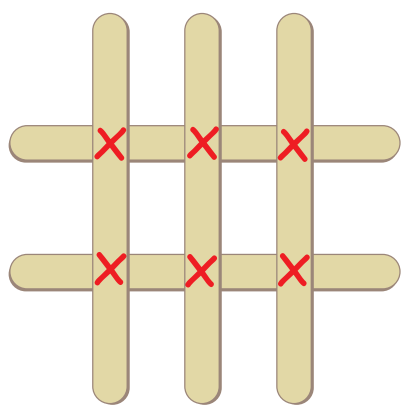
This approach is appropriate for beginning multiplication basic number facts, but is not efficient for larger multiplication number facts such as 9 x 6. The large number of rows and columns make counting the total number of entries in the array difficult. A reasonable approach would be to make 24 entries the maximum when using the array technique.

Cross-product Technique

A third way to demonstrate the concept of multiplication is the cross-product technique. This technique uses horizontal and vertical segments, either drawn or as manipulatives, such as 5" sticks. To indicate the first factor of the multiplication number fact, the learner lays the sticks horizontally. The second factor is indicated by sticks laid vertically over the others. For example, 2 x 3 would look like this:

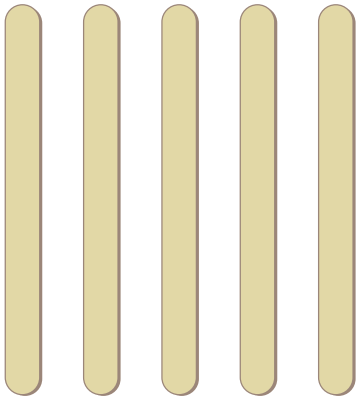


Learners count the intersections (crosses), which generate the value of 2 x 3, or 6.

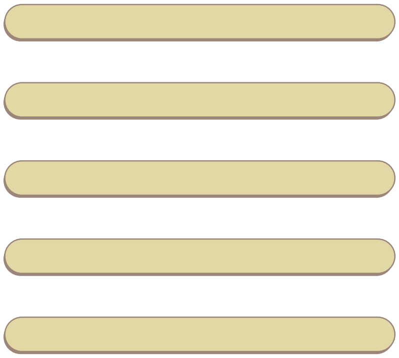


Again, you can rotate the segments 90° to generate 3 x 2 (three horizontal segments and two vertical segments). In this case, too, it is best not to go above 24 entries using the cross-product technique.

The cross-product technique demonstrates clearly what happens when zero is multiplied with other number values. The addition relationship and the array technique cannot demonstrate the result of any number (including zero) multiplied by zero. For instance, 0 x 5 using the addition technique would mean zero groups of five objects each. This is difficult to explain to primary-age learners. Demonstrating 0 x 5 using the array technique also doesn't work because it is not possible to make an array with zero rows and five columns. By contrast, the cross-product technique can demonstrate all possible combinations of zero and any number. For example, 0 x 5 is shown as follows:



The number of crosses is zero. For 5 x 0, the cross-product technique is shown in this way:



The number of crosses is zero. For 0 x 0, the cross-product technique is "shown" by zero horizontal segments and zero vertical segments resulting in zero crosses.

Math Balance

The math balance demonstrates the concept of multiplication by using the relationship of addition: 2 x 3 would be shown by 3 + 3, or 6. The math balance protocol in this section offers a more detailed explanation of finding beginning multiplication number facts. In the next section, *Extending Numerical Skills*, you will find a more complete listing of ways to demonstrate the concept of basic multiplication number facts.

Division

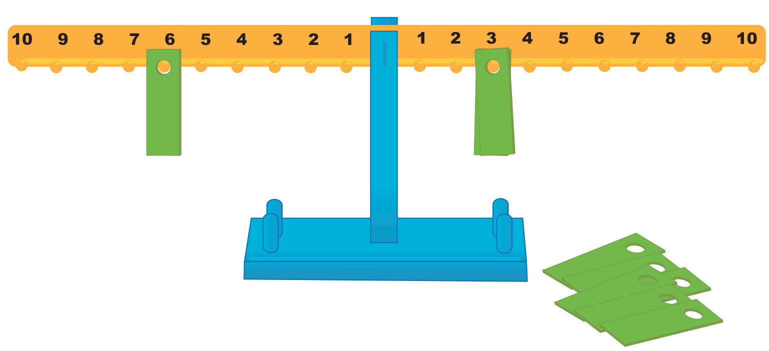
As with subtraction, the division concept is often harder for some primary-age learners to grasp in the abstract format. This is so even though children just starting school seem to have a much more elaborate construct of the concept of division than of any of the other three arithmetic operations. Many experiences in the preschool years demonstrate the division concept; young children are often reminded to share toys, books, and other objects with others. Young learners, therefore, usually understand the concept of sharing (division). However, these informal experiences do not seem to transfer to the formal, abstract mathematical division concept.

When beginning division concepts are presented, a special abstract language of specific symbols is introduced, and this may lead to confusion. The common division symbol is ÷ . The other symbols for division, — (fraction bar) and $\overline{)\quad}$ (long division symbol), have inherent difficulties for students in completely understanding their relationship to division.

Division and multiplication are inverses of each other, and this naturally yields the most common approach to introducing the concept of division—through its relationship to multiplication. For example, 6 ÷ 3 can be demonstrated by the related multiplication fact of 3 x [ ] = 6. This is the “missing factor” method. What, when multiplied by 3, will yield 6?

Division and subtraction are also aligned in that division can be shown by using repeated subtraction. For example, 18 ÷ 6 is represented by 18 – 6 = 12, 12 – 6 = 6, 6 – 6 = 0, or 18 minus 3 groups of 6 with no remaining values to subtract.

The math balance demonstrates the concept of division in terms of multiplication. The division number fact 6 ÷ 3 would be shown by using a weight on the 6 peg and two weights on the 3 peg to achieve a balance. The math balance protocol in this section expands on how to find beginning division number facts using this method.



Informal Diagnostic Strategies

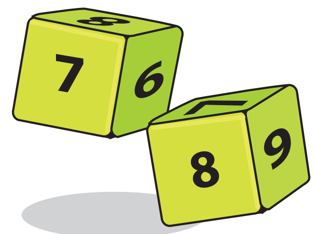
When young learners begin the process of understanding the concepts of addition and subtraction they can misunderstand or make incorrect assumptions. These conceptual errors can become barriers to the successful recall of specific addition and subtraction number facts. An information paradigm can help to diagnose and hopefully correct these misunderstandings before they lead to a more serious situation. This information diagnostic cycle starts when the teacher **detects an error** pattern in a learner's recall of an addition or subtraction number fact. The next stage is the most crucial and is the **diagnostic stage**. After determining what error may be occurring, the teacher can **prescribe specific action**. The young learner works through the exercise and the teacher **assesses the results** and **evaluates** whether the misunderstanding has been corrected. If the misunderstanding still exists, the cycle is repeated with new data.

Classroom teachers have noted that addition and subtraction misunderstandings often involve combinations of the 6s, 7s, 8s, and 9s. When this occurs, the young learner should be asked carefully designed questions to identify the exact error (step 2, diagnose). For example, the learner may recall incorrectly that 6 + 7 is 14. When asked to show how he or she arrived at the response, the learner counts on from 6 but begins the count with the other addend (7), counting out 7 to result in 14 for the response. If the misunderstanding occurs when the statement 6 + 7 is written in the abstract mode on a flashcard, for instance, a reasonable strategy would be to return to the principle **touch it, see it, think it** as reflected in the framework of hands-on materials, then pictorial representation, leading to the abstract mode. A general principle of designing a prescription is to provide intervention that is one move back from where the misunderstanding occurred. So, if the learner stumbles in the abstract mode,

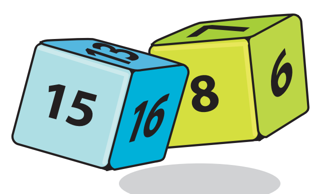
> A general principle of designing a prescription is to provide intervention that is one move back from where the misunderstanding occurred.

move back to the pictorial mode, and then double-check understanding in the hands-on mode. If in the pictorial mode, move back to the hands-on mode. In this case, the teacher might prescribe that the young learner draw six objects, draw seven more objects, and then count the total number of objects. To reinforce counting on from the first addend, the young learner should then construct the two groups with counters or other physical objects and count them, looking back at the drawing to verify accuracy. Finally, the learner would view the abstract mode to respond that 6 + 7 is 13.

Since specific addition number facts seem more likely to be misunderstood, any game or activity should employ combinations of 6s, 7s, 8s, and 9s more frequently than other combinations. For example, start with two number cubes (hexahedrons with numerals rather than dots) marked on the six faces with 6, 7, 8, and 9 plus two other numerals (could be two numerals from 6, 7, 8, 9). Roll the two number cubes and add the face values for the result. Note that using regular dot cubes is not helpful in learning the basic number facts since the young learner must recall the addition of numbers (e.g., 2 + 3) and not the addition of • • + • • •.



Classroom teachers tell us that basic subtraction number facts give rise to more errors than basic addition number facts do. However, most of the same values that were difficult within the addition number facts are also difficult within the subtraction number facts. Some young learners incorrectly recall the 17, 16, 15, and 13 combinations. Use the diagnostic cycle to clear up these misunderstandings. At the prescription stage you might construct a number-cube activity for these situations. In this activity, the faces of one cube have the single digits 6, 7, 8, 9 plus two other numerals and the faces of the other cube have 13, 15, 16, 17 plus two other values (such as 14 and 19). The learner rolls the two number cubes and subtracts the single digit from the double digit for the response.



The informal diagnostic strategy of detection, diagnosis, prescription, assessment, and evaluation can yield successful results for recalling basic addition and subtraction number facts (and may be used in content areas other than mathematics as well). Most misunderstandings at this stage of mathematical development are more readily corrected if addressed quickly. This gives young learners confidence, which translates into success in efficiently recalling the basic addition and subtraction number facts.

Strategies for Building Efficient Recall

There are many valid strategies for developing the efficient recall of addition and subtraction number facts. A common approach with primary learners is to limit the recall to number facts up to 10, at first. Later, after learners become confident in recalling these beginning addition facts, they move on to number facts up to and including 18.

Another common strategy is to start with the doubles (i.e., 1 + 1, 2 + 2, 3 + 3, 4 + 4, 5 + 5, 6 + 6, 7 + 7, 8 + 8, and 9 + 9). Once young learners are confident in recalling the doubles, they practice recalling doubles plus 1: 1 + 2, 2 + 3, 3 + 4, 4 + 5, 5 + 6, 6 + 7, 7 + 8, 8 + 9, and their commutative friends, or "turn-arounds," 2 + 1, 3 + 2, 4 + 3, 5 + 4, 6 + 5, 7 + 6, 8 + 7, and 9 + 8. Next, learners tackle the combinations of addition facts that result in 10: 1 + 9, 2 + 8, 3 + 7, 4 + 6, and their turn-arounds 9 +1, 8 + 2, 7 + 3 and 6 + 4. The remaining basic addition number facts can be practiced in two stages, first those combinations that have a result less than 10, followed by those with a result more than 10 and up to 18.

While developing the basic addition number facts, the related subtraction number facts should be learned for recall as well. Just as understanding the concept of subtraction is harder than addition for young learners, the subtraction number facts seem to take them more time to learn. Teachers should not expect learners to master the efficient recall of basic subtraction number facts until many months after they have become proficient in basic addition number facts. The following table provides a guide to reasonable expectations for addition and subtraction number facts concept development, practice recall at a rate of 3 or 4 seconds, and automaticity recall at a rate of 2 or 3 seconds.

School Level	**Concept Development**	**Practice Recall Rate: 3–4 seconds**	**Automaticity Recall Rate: 2–3 seconds**
early grade 1	addition to 10		
mid-grade 1	subtraction from 10	addition to 10	
late grade 1	addition to 18 subtraction from 18	subtraction from 10	
early grade 2		addition to 18	
late grade 2		subtraction from 18	addition to 18
late grade 3			subtraction from 18

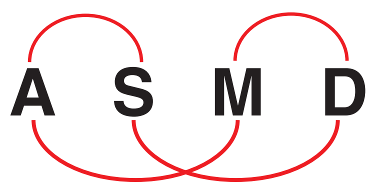
These are broad expectations that will not align with all students, e.g., special needs, ELL, and gifted students. However, these benchmarks are reasonable expectations for most students. A variety of card games, number-cube activities, and board games can be designed for students at these grade levels to practice recalling basic addition and subtraction number facts.

The strategies and expectations for the efficient recall of basic multiplication and division number facts are explained in the next section, *Extending Numerical Skills*. Third grade is a pivotal grade level for recalling addition, subtraction, and multiplication number facts. Therefore, both Section 1, *Early Numerical Skills*, and Section 2, *Extending Numerical Skills*, include strategies and activities for third-grade learners. Some of the protocols in both sections may be appropriate for third-grade learners as well.

Tips for Success

Here is a list of reminders for recalling the basic number facts. These tips are not exhaustive; they are a starting point for teachers to expand upon with the appropriate application for the given learning environment.

- Check if the young learner can count on, count back, and count after a pause upon entering first grade.
- Keep the mode sequence as the keystone principle of instruction: **hands-on to pictorial to abstract**, or **touch it to see it to think it**.
- Develop the concept, check understanding of the concept, and then introduce the recall of basic addition and subtraction number facts.
- Build the process from the interrelationship of addition, subtraction, multiplication, and division.



- Design fun activities that motivate the young learner to practice the basic addition and subtraction number facts.
- Play recall of basic number fact games that involve both strategy and chance.
- Practice the basic addition and subtraction number facts in the context of daily life.
- Have students practice recalling the basic addition and subtraction number facts as a “sponge” activity to soak up time while they are lining up or waiting for the next activity or class to start.
- Mix the easy basic addition and subtraction number facts with the harder number facts to build confidence and success.
- Remember that mathematics is the science and language of patterns. Look for patterns!

Protocols

A successful numeracy development program requires teachers to monitor students’ progress during practice, and to intervene between the practices when needed. These interventions are briefly described in the following protocols. The protocols provide teachers a basic framework for action, listing the specific topic, pre-assessment information, appropriate age level, sequence of actions, an assessment strategy for evaluation, and suggestions for adaptations or modifications for English Language Learners (ELL) as well as special needs and gifted learners.

The protocols are designed to take approximately 20 to 30 minutes. The “Launch” section may take 5 minutes, the “Explore” section 15 minutes, the “Closure” section 2 minutes, and the “Assessment and Evaluation” sections together another 5 minutes. Of course, the protocols are quite flexible in how they can be most effectively utilized in a learning situation. The timeframe of each protocol can easily be adapted to meet the learning environment.

For Section 1, *Early Numerical Skills (Grades 1–3)*, the protocols will address the concepts of addition and subtraction. Some of the protocols are games played by two learners and some are activities that involve the entire class or small groups of learners. Most of the protocols require minimal materials such as counters, simple game boards, number cubes, and number pieces. You may need to create some of these, but they are very simple. For instance, number cubes, which show numerals on the faces, can be easily made by taping numerals over the dots on dice. However, some protocols require a device called a math balance. This low-cost device is readily available, and it has the added benefit of being applicable to all four arithmetic operations.

The protocol matrix below gives the basic information of title, topic, age level, and instructional mode.

Protocol Matrix			
Title	**Topic**	**Level**	**Instructional Mode**
Sheep in the Pasture	classification, one-to-one correspondence, counting	early primary	whole group
Cover Your Facts	beginning addition facts	primary	small group
Lucky Number 7 (Less, More, Equal)	sums to 12	later primary	small group
Getting Bigger	adding by counting on	primary	small group
Target 15	addition with a strategy (sums to 20)	later primary	small group
Mystery Box	subtraction	early primary	small group
Reach In	subtraction with removal	early primary	whole or small group
Beginning Number Pieces	beginning addition	primary	whole or small group
Number Pieces	subtraction with missing addend	primary	whole or small group
Partition	subtraction with partition	primary	whole group
Addition Balance	sums to 18	primary	small group
Balancing Your Addends	subtraction with missing addends	primary	small group

Sheep in the Pasture

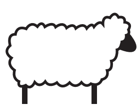
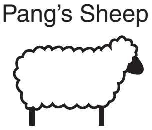
Topic: Classification, one-to-one correspondence, counting
Instructional Mode: Group
Level: Early primary
Time: 20–30 minutes

Pre-assessment
Check for ability to follow specific directions that involve a sequence of events.

Materials
- Two large green "pastures" made of felt or two large green "pastures" done electronically
- Paper or electronic images of sheep (at least nine)
- A frame or table with two rows and at least six columns

Launch
- Make two large pastures and put 5 sheep in one pasture and 4 sheep in the other. Label the pastures. Do not arrange the sheep in any pattern; place them randomly in the two pastures.
- Make a frame as shown, labeling the rows with the pasture names.

Holding pens

Jena's Sheep 					
Pang's Sheep 					

- Explain that in this activity learners will put the sheep into their holding pens at night to help determine which pasture has more sheep.

You might say, "We are going to move the sheep from their pastures into their own personal holding pen. We want to know which pasture has more sheep."

Explore
- Remind students that we want to find out which pasture has more sheep.
- Have a child move one sheep from Jena's pasture to the leftmost holding pen for Jena's sheep.
- To show one-to-one correspondence, have another child move a sheep from Pang's pasture to the leftmost holding pen for Pang's sheep.
- Have students repeat this procedure, placing sheep in the leftmost available pen and alternating pastures, until both pastures are empty.

- When finished, ask the students to name the pasture that had more sheep. (The frame shows which pasture had more sheep.)
- Repeat with other combinations of sheep in each pasture.

Later in the year learners could count the sheep placing them all in the holding pens.

Closure

- Ask learners again how they found out which pasture had more sheep.
- Have two children put the sheep from the holding pens back into the appropriate pastures. Repeat for a final check.

Assessment	***Evaluation***
Have various children move the sheep between the holding pens and the pastures.	Success will be accurate classification, one-to-one correspondence, and counting.
Observe their ability to conserve the one-to-one correspondence and classification.	

Adaptations for Various Learners

ELL:

- Use animals native to other countries.
- Explain in detail the directions of the movement of the sheep from the pasture to the holding pen.

Special Needs:

- Make easy-to-grasp animals.
- Make the frame or holding pens large and easy to access.
- Use smaller numbers of sheep in the beginning, e.g., 2 and 3.

Gifted:

- Ask students to count the sheep.
- Use larger numbers of sheep.
- Use four pastures with different numbers of sheep.

Cover Your Facts

Topic: Beginning addition facts
Instructional Mode: Small group
Level: Primary
Time: 20–30 minutes

Pre-assessment
Check for numeral recognition and understanding of doubles.

Materials
- Number cubes with the numerals 1, 2, 3, 4, 5, and 6 on faces
- Counters (at least 36)
- A Double Cover board and an Uncover board
  (You will need to create these boards. They are shown below.)

Launch
- These two board games each involve two players.
- Double Cover requires 1 number cube and 12 counters.
- Uncover requires 2 number cubes and 24 counters.
- Double Cover gives practice in doubles.
- Uncover gives practice in addition.
- Explain that in this activity learners will take turns with a partner practicing with doubles and with addition.

Explore
Here's how to play the games.

Double Cover
Board

2	**4**	**6**	**8**	**10**	**12**
2	**4**	**6**	**8**	**10**	**12**

Directions
- Two learners sit on opposite sides of the board.
- They take turns rolling one number cube.
- When a 1 is rolled, the player doubles it and covers the 2 with a counter on his or her side of the board. When a 6 is rolled, the player doubles it and covers the 12 with a counter.
- If the number is already covered, it is the next player's turn.
- The game ends when one player covers all six numbers.

Uncover
Board

2	3	4	5	6	7	8	9	10	11	12
2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**

Directions

- Two players sit on opposite sides of the board.
- They place a counter under each numeral.
- Players take turns rolling the two number cubes.
- The player rolls the number cubes and recalls the sum rolled. The sum value counter is removed.
- If the counter has already been removed, the turn goes to the other player.
- The game ends when one player removes all the counters on his or her side.

Closure

- Review the meaning of double in addition number facts.
- Review the addition facts of 1 to 6.

Assessment	***Evaluation***
Check if directions are followed correctly.	—
Spot-check the understanding of doubles.	Success is the ability to quickly recall doubles of 1, 2, 3, 4, 5, and 6.
Spot-check the understanding of sum, or addition of two values.	Success is the ability to accurately add the values of the two number cubes.

Adaptations for Various Learners

ELL:

- Demonstrate the meaning of the word double with physical objects.
- Review the meaning of the words sum and addition with physical objects.

Special Needs:

- Modify the Double Cover board to include only 2, 4, and 6.
- Modify the Uncover board to include only 2, 3, 4, 5, and 6.
- Modify the number cubes so that they have only 1, 2, and 3 on the faces.

Gifted:

- Modify the Double Cover board to do doubles + 1.
- Modify the Uncover board to show 12, 13, 14, 15, 16, 17, and 18. Have the two number cube faces contain only 6, 7, 8, and 9

Lucky Number 7 (Less, More, Equal)

Topic: Sums to 12, concepts of less than and greater than
Instructional Mode: Small group
Level: Later Primary
Time: 20–30 minutes

Pre-assessment
Check understanding of less than, more than, and equal to a number.

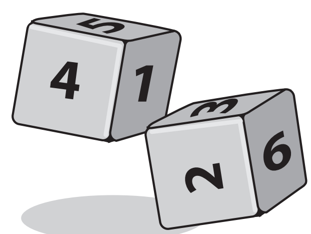
Materials
Two number cubes with faces 1, 2, 3, 4, 5, and 6

Launch
Lucky Number 7 is a game of chance that uses players' addition skills as well as the concepts of less than, equal to, and more than.

- Two or more children can play this game.
- A player will choose a target sum (between 2 and 12), state it aloud and say whether it is less than, equal to, or more than 7.
- The player rolls the number cubes and finds the sum. If the choice is less than or more than 7 and the player rolled his or her choice, the player receives 1 point.
- If the target sum is not rolled, the player does not get any points.
- If the target sum is 7 and the player rolled a 7, the player receives 5 points.

Explore
Let learners play the game and find out what happens. Here's a sample of how it works:

- The first child declares that the sum of the two number cubes will be 6 [less than 7].
  The child rolls a 9 [6 + 3] and receives 0 points.
- The next child declares that the sum of the two number cubes will be 8 [more than 7].
  The child rolls an 8 [5 + 3] and receives 1 point.
- The next child declares that the sum of the two number cubes will be 7 [equal to 7].
  The child rolls a 7 [4 + 3] and receives 5 points.
- The game ends when one child reaches 20 points.



Closure

- Discuss how often the sum of 7 occurs during the game.
- Discuss whether there should be a different number than 7 as the less than, equal to, and more than value.

Assessment	Evaluation
Check accurate addition number facts using the number cubes.	—
Check understanding of less than, equal to, and more than concepts.	Successful decision of when a sum is less than, equal to, or more than 7.

Adaptations for Various Learners

ELL:

- Show less than, equal to, and more than with physical objects.

Special Needs:

- Change the rules to use only less than (or more than) for points.

Gifted:

- The child decides the break value (replacing 7).
- The child decides the scoring system, e.g., *more than* is 5 points, *less than* is 2 points, and *a miss* means the player loses 3 points.

Getting Bigger

Topic: Adding by counting on
Instructional Mode: Small group (2 learners)
Level: Primary
Time: 20–30 minutes

Pre-assessment
Check understanding of counting on from a larger number (5 or above) to a specific number.

Materials
4 number cubes (traditional cubes with the faces covered by different values)

- 2 number cubes with faces 1, 2, 3, 4, 5, and 6
- 2 number cubes with faces 4, 5, 6, 7, 8, and 9

Launch

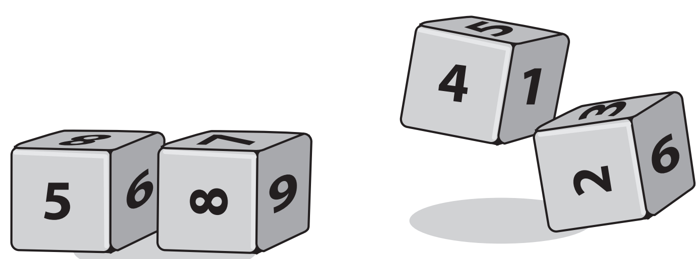
- Roll two number cubes. Say the faces show 3 and 5. Using the larger value (5), count on by saying "six, seven, eight."
- Write the addition equation 5 + 3 = 8 to reinforce the basic addition number fact.

Explore

- This activity is appropriate for pairs of learners.
- Each pair of learners receives either two number cubes with 1–6 or with 4–9 on their faces. It is possible to have one number cube from each group as well, that is, one from the 1–6 set and one from the 4–9 set.
- One child rolls the two number cubes. If, for example, the child rolls 6 and 9, he or she starts with the larger value (9) and counts on six more to reach 15.
- The other child must write the correct addition equation, 9 + 6 = 15.
- Have the learners change roles and do the activity several more times. Changing roles each time will reinforce both counting on and writing the correct addition equation.

Closure

- Within the pairs of learners have one child declare a number (e.g., 7) and the count-on number (e.g., 6). The other child must count on 6 more from 7.



Assessment	Evaluation
Roll a 1–6 number cube and a 4–9 number cube, and have the child perform the count-on activity. For example, if the numbers 8 and 3 are thrown, the child should count on 3 more than 8.	Success is verified by counting "9, 10, 11" and writing the equation 8 + 3 = 11.

Adaptations for Various Learners

ELL:

- Demonstrate counting on from various numbers, e.g., start at 7 and count on by 6 more.
- Demonstrate counting on by using physical objects and moving them to form a group. For example, show 5 objects and count on by 2 more. Group the 5 objects together, and count them out; then take two objects and count on "6, 7." Show the total group now as 7 objects.

Special Needs:

- Demonstrate the action of counting one.
- Begin with two groups that result in less than 10 when joined together. Use number cubes with faces 1, 2, 3, 3, 4, and 5.
- Later expand the values of the two groups. Use the number cubes with 1–6 values on the faces.

Gifted:

- Practice counting on from the larger value with the two number cubes 1–9.
- Then roll the two number cubes 1–9. Rather than count on, count *back* from the larger number using the value showing on the other number cube. For example, if 8 and 5 are rolled, start at 8 and count back "7, 6, 5, 4, 3" or 8 – 5 = 3.
- Switch between counting on and counting back every other roll.

Target 15

Topic: Addition with strategy (sums to 20) **Level:** Later Primary
Instructional Mode: Small group **Time:** 20–30 minutes

Pre-assessment
Check understanding of addition to 20.

Materials
4 number cubes (traditional cubes with the faces covered by different values)

- 2 number cubes with faces 0, 1, 2, 3, 4, 5
- 2 number cubes with faces 5, 6, 7, 8, 9, 10

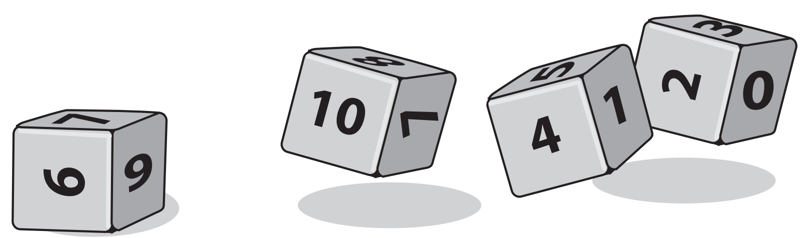
Launch
Two learners participate. The object is to roll a sum as close to 15 as possible.

Explain the directions of this activity and/or demonstrate as follows:

- The first learner chooses any two number cubes to roll and finds the sum of the numbers. For example, 5 and 3 are rolled; the sum is 8.
- The learner may decide to stop or to roll a chosen third number cube. For example, if the number rolled is a 9, the sum is 17. A result larger than 15 is valid.
- Now the second learner rolls the number cubes following the same directions. After three number cubes are rolled, suppose the second learner's sum is 11. This learner can either stop or roll the fourth number cube. The learner will choose the strategy—roll again or stop—that he or she thinks will result in a sum closer to 15.
- The learners then decide who is closer to the Target 15.

Explore

- Have pairs of learners do the Target 15 activity for seven or eight rounds.
- Ask the pair of learners about their strategy in deciding which number cube to choose and when to stop rolling the number cubes.
- Have the learners record their values and final addition equations, for example, 5 + 3 = 8 and 8 + 9 = 17.



Closure

- Check for understanding of which number cube is most appropriate in specific situations.

Assessment	Evaluation
Set up a pair of number cubes so that 11 is the sum. Ask learners which number cube (0–5 or 5–10 number cube) would be an appropriate choice to reach a sum close to 15.	Success is the choice of the 0–5 number cube.

Adaptations for Various Learners

ELL:

- Using diagrams, demonstrate the different choices of number cubes.
- Have the learner record each number as the number cubes are rolled.

Special Needs:

- Use only one 0–5 number cube with only one 5–10 number cube at the beginning.
- Include another 0–5 number cube after success with the two number cubes.

Gifted:

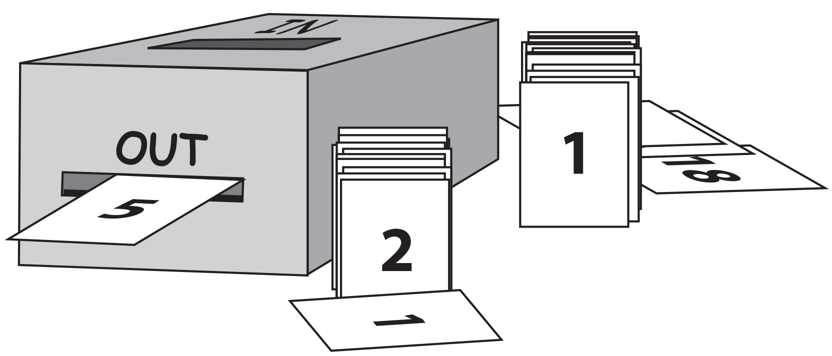
- Have the learners determine which number cube is the best to roll first.
- Have the learners demonstrate a successful strategy that seems to yield sums close to the Target 15.

Mystery Box

Topic: Subtraction
Instructional Mode: Small group (two learners)
Level: Early Primary
Time: 20–30 minutes

Pre-assessment
Check understanding of the concept of subtraction as a take-away operation.

Materials
- Box with open back and two slots, one marked IN and the other marked OUT
- Two sets of statement cards with the values 1 through 18



Launch
This is a game of "take away." One child determines a take-away rule for each session. The other child (tester) tests the rule.
- Demonstrate how the game is played. Tell the children you have a take-away rule in mind. Have in mind the rule of "take away 3," for example. Hand one child a card to start and say, "You'll be the tester." The child inserts the card in the IN slot. Let's say the card has the number 8 on it. Return a 5 card to the child through the OUT slot. Ask the child to name the "take away" rule or to ask for another card to put IN.
- Emphasize that the mystery box always follows the rule. The tester can always ask for another card to put in the IN slot to test and see what the take-away rule is.
- Tell the children one of them will have a rule in mind and always follow the take-away rule, as you did. The other will test the rule in the mystery box: "See if you can discover the rule."

Explore
- A pair of learners decides who makes the rule and who tests the rule.
- The rule cannot be shared until later.
- Learners play a session to become familiar with how the rule and mystery box works.
- Continue playing until the child who is the tester thinks he or she knows the rule.
- Change roles.

Closure

- Select a rule, for instance, take away 4.
- Insert a 12 card in the IN slot.
- Pull an 8 card out of the OUT pile.
- Repeat a few times with different starting values and find the card with the corresponding result following the rule.
- Ask learners, “What do you think the rule is?”

Assessment	*Evaluation*
Show learners this chart and ask, “What do you think the rule is?” IN 8 OUT 6 IN 9 OUT 7 IN 3 OUT 1 IN 6 OUT 4	Success is declaring that the rule must be “take away 2.”

Adaptations for Various Learners

ELL:

- Demonstrate the mystery box with specific values.

Special Needs:

- Demonstrate the mystery box with specific values.
- Use statement cards with values 1 through 9.
- Later include some values larger than 9.

Gifted:

- Have the child make a more complex rule, for example, “take away 3 and add 4.”
- Have the child explore a rule that doubles the number (6 becomes 12, 7 becomes 14, and 8 becomes 16).

Reach In

Topic: Subtraction concept using removal
Instructional Mode: Whole or small group
Level: Early Primary
Time: 20–30 minutes

Pre-assessment
Check for understanding of the concept of counting and early subtraction with removal or take away.

Materials
- Paper or opaque cloth bag
- 18 counters

Launch
Demonstrate the meaning of removing counters, and how to do this activity.
- For example, in view of the children, place 15 counters in the bag so that they know how many are there.
- Reach into the bag and remove 6 counters.
- Have a child predict how many counters remain in the bag.
- After the prediction, count the counters in the bag to check the subtraction fact 15 – 6 = 9.
- Have the child record the equation: 15 – 6 = 9.

Explore
This activity is appropriate for 2 to 4 learners per bag. Here's how learners perform the activity. Stress the importance of doing them in sequence.
- Put a known number of counters in the bag.
- Reach in and remove any number of counters.
- Record this number.
- Predict the number of counters remaining in the bag.
- Record this number.
- Finally, write the subtraction equation.

Closure

- Review the meaning of removal by removing counters from the bag.
- After the first exploration with a specific number of counters is completed, change the number of counters in the bag and repeat through closure. (Players may take turns determining the number of counters to place in the bag, or the teacher might determine this number to practice specific basic subtraction number facts.)

Assessment	*Evaluation*
Construct a situation with 12 counters in the bag, remove 3 counters, and ask how many are remaining in the bag.	Success is the ability to predict accurately the subtraction fact of 12 – 3 = 9.
Check for subtraction understanding using removal with the written equation, 12 – 3 = [ ] or 12 – 3 = 9.	Success is the ability to write the correct subtraction equation.

Adaptations for Various Learners

ELL:

- Demonstrate the action of removal using a diagram or through physical movement.
- Demonstrate how to write the subtraction fact equation.

Special Needs:

- Demonstrate the action of removal using a diagram or through physical movement.
- Demonstrate how to write the subtraction fact equation.
- Modify the total number of counters in the bag to 9.
- Later adjust the total number of counters as appropriate for the learner.

Gifted:

- Use another bag with the values 5 through 18 recorded on small pieces of paper, and have the child draw out the number of counters to place in the bag.
- After the “Reach In” equation is found, have the child determine all the possible equations using those numbers. For example, after finding 12 – 3 = 9, have the child write down all possible subtraction equations beginning with 12 counters in the bag.

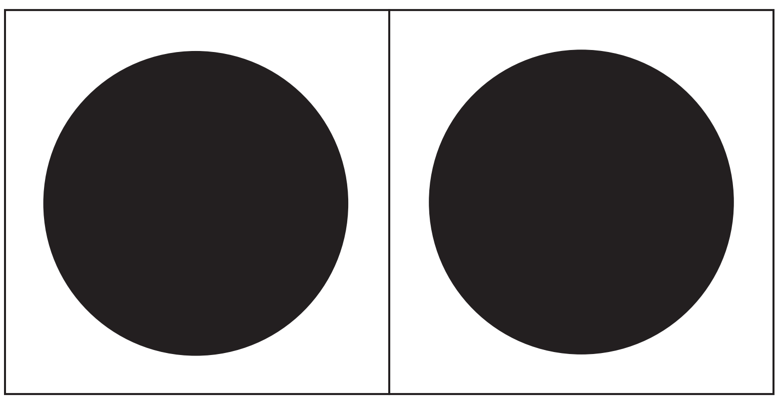
Beginning Number Pieces

Topic: Beginning addition | **Level:** Primary
Instructional Mode: Whole or small group | **Time:** 20–30 minutes

Pre-assessment
Check for understanding of the concept of addition as a joining together of two groups.

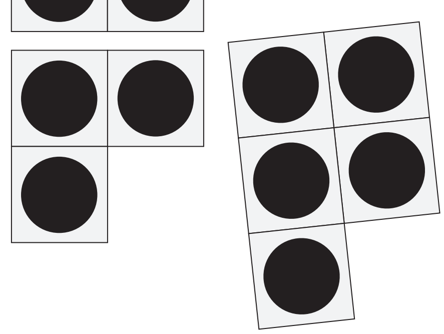
Materials
- A set of number pieces from 1 to 10 (see Appendix, pages 109-116, for reproducible forms)

Example: 2 number piece (actual size)



- Index cards with individual addition statements, with sums up to 18, such as 4 + 5 = [ ].

Launch



- Show how to form a number piece for 3 + 2 = [ ]. Take the 3 number piece and the 2 number piece and place them together like a jigsaw puzzle. Find the number piece in the pile that will cover the 3 number piece and the 2 number piece exactly (5 number piece).
- Demonstrate how the 3 number piece and the 2 number piece can be joined together in another configuration and still be covered by the 5 number piece. This shows the commutative property of the addition operation.
- Demonstrate how to use the 10 number piece to construct values larger than 10. For example, a 15 number piece can be formed by combining the 10 number piece and the 5 number piece.

Explore

- Give learners statement cards involving values less than 10, such as these:
  - 6 + 3 = [ ]
  - 7 + 1 = [ ]
  - 4 + 2 = [ ]
- Ask them to join the pieces and then find the number piece that covers the two joined number pieces exactly.
- When learners are comfortable with the lower sums, hand them statement cards that involve two-digit results such as these:
  - 8 + 7 = [ ]
  - 9 + 5 = [ ]
  - 6 + 5 = [ ]
- Later the two-digit and the single-digit addition statement cards can be combined for practice and review.

Closure

- Check for understanding of this concept of addition: joining together of two groups for a result.
- Review a single-digit statement and a double-digit statement using the number pieces.

Assessment	*Evaluation*
Have the child demonstrate the following statements using the number pieces:	
6 + 9 = [ ]	Success is verified when the child finds the 10 number piece and 5 number piece or any other combination that makes the sum of 15.
6 + 7 = [ ]	Success is verified by finding the 10 and 3 number pieces for 6 + 7 = [ ] (13).

Adaptations for Various Learners

ELL:

- Use diagrams to show how to use the number pieces to join two number pieces for a result.

Special Needs:

- Use diagrams to show how to use the number pieces to join two number pieces for a result.
- Start with simple one-digit addend statements, including doubles and doubles + 1.

Gifted:

- Have the young learner explore other possible solutions using the number pieces.
  For example 8 + 4 = [ ]. One possible result could be the 10 number piece and the 2 number piece. However, the 7 number piece and the 5 number piece also cover the 8 number piece and 4 number piece. Have the child find all possible number pieces that cover 8 + 4 = [ ].

Number Pieces

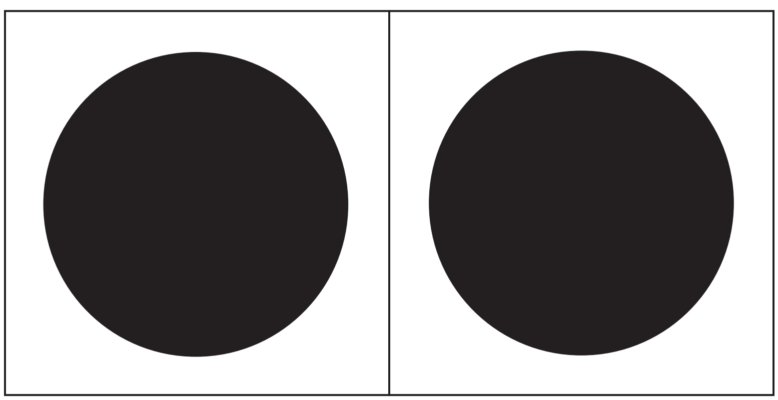
Topic: Subtraction with missing addend
Instructional Mode: Whole or small group

Level: Primary
Time: 20–30 minutes

Pre-assessment
Check for understanding of the concept of subtraction with removal or take away.

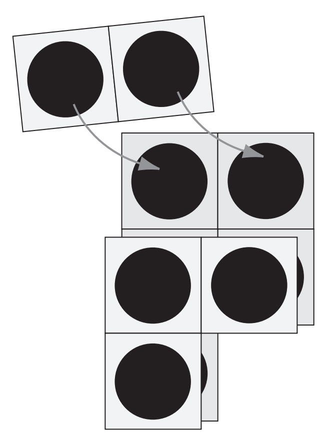
Materials
- A set of number pieces from 1 to 10 (see Appendix, pages 109-116, for reproducible forms)

Example: 2 number piece (actual size)



- Index cards with missing addend statements with values up to 18, such as 5 = 3 + [ ], [ ] + 8 = 16. You can use the same statements constructed for Balancing Your Addends, pages 48-49.

Launch



- Show learners how to cover the number piece to demonstrate 5 – 3 = [ ] or 5 = 3 + [ ]. Take the 5 number piece, find the 3 number piece, and place it over, appropriately, to cover 3 dots on the 5 number piece. Find the number piece needed to cover the remaining two dots (2 number piece).
- Demonstrate how to use the 10 number piece to construct values larger than 10. For example, the 15 number piece is formed by placing the 10 number piece and the 5 number piece together.

Explore

- Using index cards with statements involving values less than 10, learners complete the covering of the number piece to find the correct response. For example:
  - 5 = 3 + [ ]
  - [ ] + 4 = 6
  - 9 = [ ] + 8
- Next, learners use cards with missing addend statements involving two-digit values. They cover the number pieces to find the correct response. For example:
  - 15 = [ ] + 8
  - 17 = 9 + [ ]
  - 6 + [ ] = 14
  - 6 + [ ] = 15
- Later, learners can combine the two-digit and the single-digit statement cards for practice and review.

Closure

- Check for understanding regarding the concept of the missing addend and its relationship to subtraction.
- Review a single-digit statement and a double-digit statement using the number pieces.

Assessment	***Evaluation***
Have the child demonstrate the following statements using the number pieces:	
11 = [ ] + 6	Success is verified by placing the 5 number piece correctly for 11 = [ ] + 6.
9 + [ ] = 16	Success is verified by placing the 7 number piece correctly for 9 + [ ] = 16.

Adaptations for Various Learners

ELL:

- Use diagrams to show how to use the number pieces to complete missing addend statements.

Special Needs:

- Use diagrams to show the action or movement of the number pieces.
- Start with simple one-digit addend statements, including doubles and doubles + 1.

Gifted:

- Have the young learner explore other possible solutions using the number pieces. For example, 15 = 6 + [ ]. One possible cover would be the 9 number piece, but the 4 number piece and the 5 number piece will also cover the 15 number piece. Have the child find all possible number pieces that cover the 15 number piece.

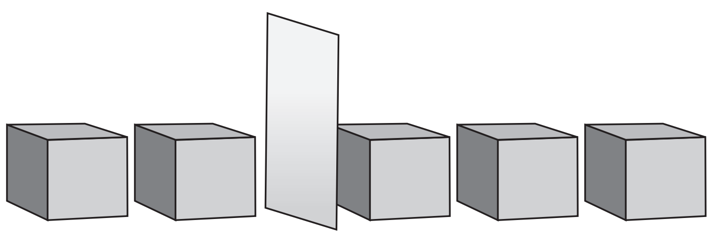
Partition

Topic: Concept of subtraction with partition
Instructional Mode: Whole group
Level: Primary
Time: 20–30 minutes

Pre-assessment
Check for understanding about separating a group into two distinct groups.

Materials
- Counters or physical objects
- Index cards to create individualized partitions

Launch
- Give each student an index card to use as a partition. (Optional: Let the student color the card.)
- Demonstrate how the partition can separate a group of five objects into two groups. For example, a group of one and another group of four; a group of two and another group of three.



- Practice placing the partition at different positions with groups of various sizes.

Explore
- Begin with a situation of three objects. Ask learners to partition this group into two and one.
- Now present the subtraction number fact of 3 – 2 = [ ]. Repeat the partitioning of 3 objects into 2 and 1.
- Reinforce the 3 – 2 = 1 result.
- Have learners practice various one-digit partitions.
- Have learners record the pictorial mode of what was just partitioned.
- Have learners record the abstract mode of what was partitioned.
- Repeat the paradigm of hands-on to pictorial to abstract.

Closure
- Discuss the technique of partitioning, or breaking one group into two groups.
- Have a learner show the partitioning of a situation such as 9 – 5 = [ ].

Assessment	Evaluation
Construct situations that show the hands-on, pictorial, and abstract modes. Have young learners match up the correct situation with the appropriate subtraction number fact.	Success will be shown by correctly matching the various mode situations to the subtraction number facts.

Adaptations for Various Learners

Special Needs:

- Construct subtraction number facts that are single digits.
- Draw a diagram of the action required to do a partition of a group of objects into two groups.

Gifted:

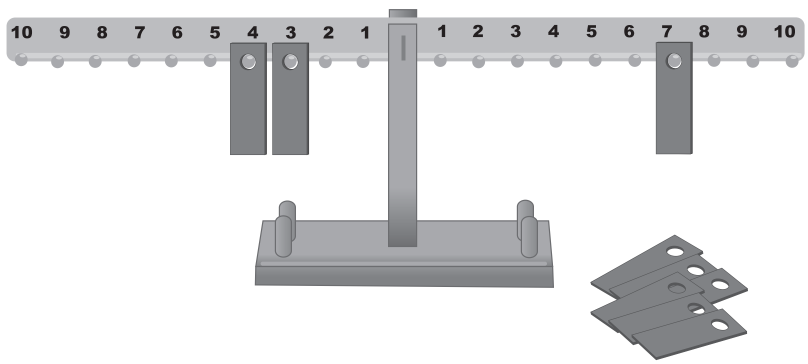
- Ask the young learner to demonstrate the various zero situations in subtraction using partitioning. For example: 5 – 0 = [ ], 0 – 0 = [ ], 5 – 5 = [ ]
- Have the young learner demonstrate various two-digit (up to 18) subtraction number facts using partitioning.

Addition Balance

Topic: Sums to 18
Instructional Mode: Small group
Level: Primary
Time: 20–30 minutes

Pre-assessment
Check for understanding of counting and beginning addition.

Materials
- Math balance device and weights



- Statements of addition up to 18, written on index cards

Launch
Demonstrate how the math balance works by putting weights on the pegs.
- Weights and distance from the fulcrum determine the correct solutions.
- Demonstrate how the sum of two addends can be found using the math balance.
  For example: 3 + 4 = [ ]
  On the left side of the math balance hang one weight on the 3 peg and one weight on the 4 peg. Have a child determine on which peg on the right side to place a weight to balance the device. It should be hung on the 7 peg. Therefore, 3 + 4 = 7.
- One math balance is sufficient for a learning center. Two children can use the math balance at the learning center.

Explore
- Learners use the math balance to find sums on statement cards involving sums less than 10, such as 3 + 2 = [ ], 6 + 3 = [ ], 4 + 4 = [ ].
- Learners next use the math balance to find sums on cards showing statements involving two-digit values such as 8 + 7 = [ ], 9 + 8 = [ ], 6 + 4 = [ ].
- Later the two-digit and the single-digit statement cards can be combined for practice and review.

Closure

- Check understanding regarding the concept of addition.
- Review a single-digit statement and a double-digit statement using the math balance.

Assessment	***Evaluation***
Have the child demonstrate the following statements on the math balance:	
6 + 5 = [ ]	Success is verified by placing one weight on the 10 peg and one weight on the 1 peg for 6 + 5 = 11.
9 + 7 = [ ]	Success is verified by placing one weight on the 10 peg and one weight on the 6 peg for 9 + 7 = 16.
7 + 8 = [ ]	Success is verified by placing one weight on the 10 peg and one weight on the 5 peg.

Adaptations for Various Learners

ELL:

- Use diagrams to show how to use the math balance to complete addition statements.

Special Needs:

- Use diagrams to show the action of the math balance.
- Start with one-digit addition statements including doubles and doubles + 1.

Gifted:

- Have the learner explore other possible solutions. For example, in the situation 6 + 9 = [ ], one weight could be on the 10 peg and another weight on the 5 peg but it is also possible to have one weight on the 8 peg and another weight on the 7 peg.
- Have the learner find all the possible weight combinations for specific basic addition facts.

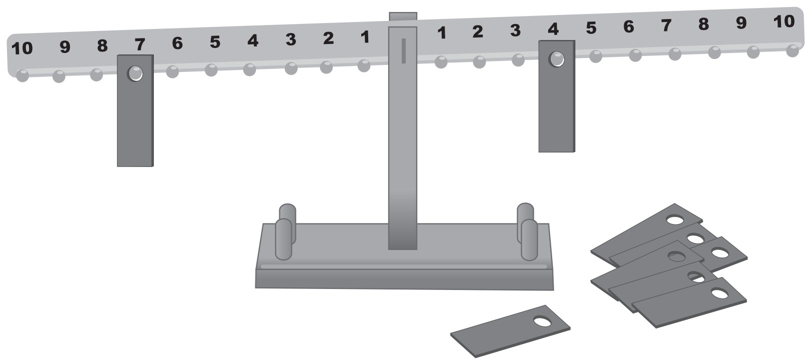
Balancing Your Addends

Topic: Subtraction concept with missing addend **Level:** Primary
Instructional Mode: Small group **Time:** 20–30 minutes

Pre-assessment
Check for understanding of the concept of subtraction with removal or take away.

Materials

- Math balance device and weights



- Statements of missing addend situations up to 18, written on individual index cards

Launch
Demonstrate how the math balance works by putting weights on the pegs.

- Weights and distance from the fulcrum determine the correct solution.
- One math balance device is sufficient for a learning center.
- Two children can use the math balance at the learning center.
- Demonstrate how the missing addend value can be found.
  For example, 7 = 4 + [ ]
  On the left side of the math balance, hang one weight on the 7 peg; on the right side of the math balance, hang one weight on the 4 peg. Have a child determine on which peg a weight should be placed on the right side to balance the device. It should be hung on the 3 peg. Therefore, 7 = 4 + 3 or 7 – 4 = 3.

Explore

- Hand out cards with statements involving values less than 10, such as 5 = 3 + [ ], [ ] + 4 = 6, 9 = [ ] + 8. Have learners use the math balance to find the missing addends.
- Next hand out cards with missing addend situations involving two-digit values such as these: 15 = [ ] + 8, 17 = 9 + [ ], 6 + [ ] = 14, 6 + [ ] = 15. Let the learners practice finding the missing addends using the math balance.

- Later the two-digit and the single-digit statement cards can be combined for practice and review.

Closure

- Check for understanding regarding the concept of the missing addend and its relationship to subtraction.
- Review a single-digit statement and a double-digit statement using the math balance.

Assessment	***Evaluation***
Have the child demonstrate the following statements on the math balance:	
11 = [ ] + 6	Success is verified by placing the one weight on the 5 peg for 11 = [ ] + 6.
9 + [ ] = 16	Success is verified by placing the one weight on the 7 peg for 9 + [ ] = 16.

Adaptations for Various Learners

ELL:

- Use diagrams to show how to use the math balance to complete missing addend statements.

Special Needs:

- Use diagrams to show the action of the math balance.
- Start with simple one-digit addend statement including doubles and doubles plus 1.

Gifted:

- Have the learner explore other possible solutions with more than one weight. For example, 15 = 6 + [ ]. One weight could be on the 9 peg but it is also possible to have one weight on the 4 peg and another weight on the 5 peg to balance the device. Have the child find all the possible weight combinations that balance the device.

Extending Numerical Skills *(Grades 3–8)*

2

Extending Numerical Skills

(Grades 3–8)

Introduction

The conceptual foundation of multiplication and division is constructed in the primary grades at the same time students learn their addition and subtraction concepts. Starting in third grade, and continuing for many years after, students will delve more deeply into multiplication and division concepts and basic number facts. In this section, we will discuss approaches to developing numeracy for students in grades three through eight.

Because addition and subtraction operations are part of the foundational knowledge needed for multiplication and division, this section begins with a brief review of addition and subtraction concepts before presenting an in-depth discussion of multiplication and division concepts. Third grade is a crucial grade in enabling learners to transition to more complex mathematics. At this juncture, a conflux of concept development must occur. That is why both Section 1 and Section 2 include third-grade activities.

Starting in fourth grade and continuing through eighth grade, students will more completely develop their conceptual understanding of multiplication and division, and will also focus on mathematical algorithm development. A concept, such as how we can multiply two values to result in a third value, demands abstract reasoning for complete understanding. To help students grasp concepts, teachers need to present them in more than one context, involving more than one construct and through repeated activities or lessons. In contrast, algorithm development involves procedural instruction that leads to a correct solution. Students must learn a sequence of steps and strictly adhere to it in order to reach the right answer. For example, learners can design, check for accuracy and general application, and memorize a mathematical algorithm for multiplying 2 two-digit numbers together.

In addition to concept and algorithm development, a third element in attaining numeracy is regular practice of basic number facts. The goals at this stage are for students first to become confident in multiplying and dividing, and only then to work towards automaticity of recall.

Each of these three aspects—concept development, algorithm development, and skill practice—is necessary for success in mathematics. Ideally, there should be a smooth transition from the primary grades to the intermediate grades to the middle school grades, but that does not always happen. Young learners who have not mastered their multiplication and division number facts or who have slow recall do not succeed in math in later years. Many learners struggle with the recall of basic multiplication and division number facts into the 6th, 7th, and 8th grades. Therefore, the activities and protocols described in this section may be used in grades 3 through 8, with appropriate modification.

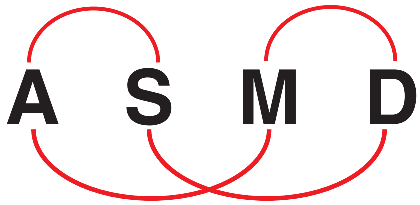
Each arithmetic operation presents its own difficulties. Many upper-elementary learners and some middle-school learners continue to derive their basic number facts. That is, they must count-on to find a sum rather than simply recall the sum as a known fact. Some of these same learners also struggle with rapid recall of the multiplication and division number facts. This section presents age-appropriate activities to assist the upper-elementary and middle-school learners achieve rapid recall, continuing to emphasize the instructional principle of **touch it, see it, think it.** The activities and protocols address the three stages of this principle through hands-on, pictorial, and abstract modalities. This section also emphasizes the importance for numeracy development of the learning sequence 1) understand the concept, 2) practice until confident, and 3) practice for automaticity.

Touch it.	See it.	Think it.
concrete	**pictorial**	**abstract**

As in Section 1, a set of protocols is included here. These protocols are designed primarily for small group and individual instruction, since whole group instruction in basic numeracy is not usually needed at the upper-elementary and middle-school levels. Some of the protocols can be modified for whole group instruction, however, if needed. The protocols present addition, subtraction, multiplication, and division activities appropriate for upper-elementary and middle-school learners. Most use materials that are readily available, such as number cubes, counters, and number statement cards. A few protocols use a device called a math balance, which was introduced in Section 1. At the end of this section you'll also find a list of ten *Tips for Success*, which are reminders of best practices in teaching the basic number facts.

Fundamental Concepts of Addition and Subtraction

Understanding the interrelationships of the four arithmetic operations, addition, subtraction, multiplication, and division is crucial to acquiring basic number facts and a solid foundation of number sense. A firm foundational understanding of addition and subtraction is also important in developing the concepts of multiplication and division because the multiplication concept is built on the concept of addition and the division concept is built on the concept of subtraction.



Addition

Addition is a binary operation that joins two addends to form a sum. The concept of addition may be described as a sliding or joining together of two groups into a single resultant group. (There are many ways to present the concept of addition. These are explained in Section 1.) Addition is a commutative operation: 3 + 2 yields the same result as 2 + 3. Students need to understand this property of addition to learn the concept of multiplication.

Subtraction

Subtraction can be described as a breaking apart of one group into two groups. There are many approaches to understanding this concept, which have been explained in Section 1. Subtraction is not a commutative operation: 5 – 3 is not the same as 3 – 5. This property of the subtraction operation will be important when developing the concept of division. While most third-grade learners will be comfortable with the addition basic number facts, a good number will struggle with subtraction number facts. Therefore, the inverse relationship of addition and subtraction must be carefully explained, investigated, and explored in third grade. With a firm understanding of addition and subtraction concepts, the learner is prepared to explore the multiplication and division concepts.

While most third-grade learners will be comfortable with the addition basic number facts, a good number will struggle with subtraction number facts.

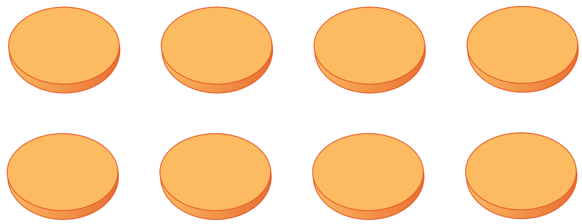
Fundamental Concepts of Multiplication and Division

Teachers introduce multiplication and division concepts after devoting a substantial amount of time to addition and subtraction number fact acquisition. However, multiplication and division are clearly aligned with addition and subtraction concepts.

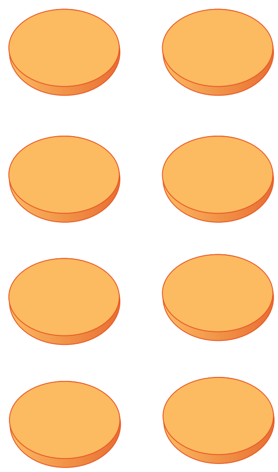
Multiplication

Multiplication can be viewed as repeated addition while division can be viewed as repeated subtraction. For example, to find the solution to 36 x 7, you could add 36 seven times or add 7 thirty-six times. Therefore, multiplication is just a short cut for doing addition. Section 1 presented three different ways to teach the multiplication concept. One approach is to exploit this addition relationship. For example, 2 x 4 could be represented by 2 groups of 4 objects, or 4 + 4, which results in 8. Since multiplication and addition are related there is also the “turn around” feature, or the commutative property: 4 x 2 produces the same result.

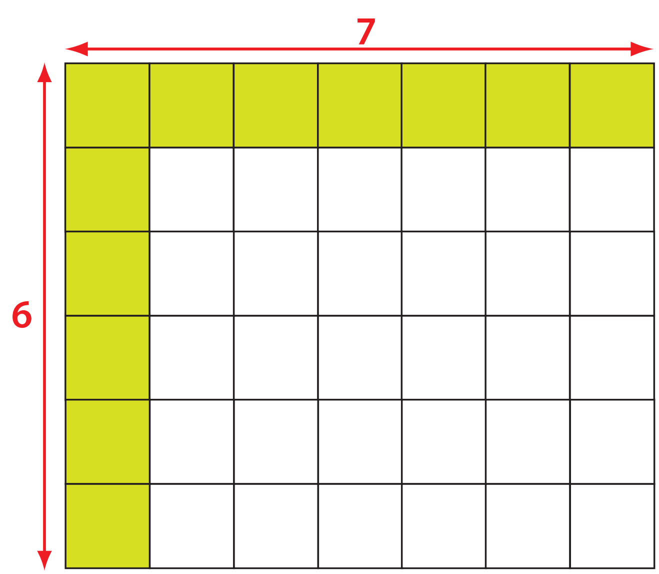
Another approach to the concept of multiplication is to use an array of rows and columns. The number of rows represents the first factor of the multiplication number fact and the number of columns, the second factor.
For example, 2 x 4 is shown in the array as follows:



This diagram can be rotated 90° to show 4 × 2.

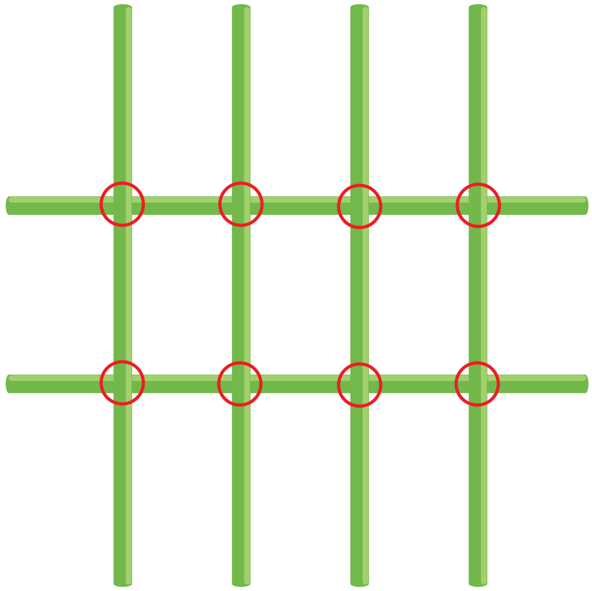


Although the array approach is limited (it becomes inefficient to count the values in the array when they exceed 24 entries), an area model, which is similar to an array, can be substituted. The area model uses the properties of length and width of a rectangle to find the number of square units in the rectangle. For example, here is how you would illustrate 6 × 7, where 6 is the width and 7 is the length:



This method provides a good visual representation of 6 × 7 (6 rows of 7 squares or 7 columns of 6 squares) and its result, 42.

Another technique to demonstrate the concept of multiplication is called cross products. Horizontal and vertical segments form a cross-product diagram. For example, 2 × 4 would be shown as follows, where 2 is the number of horizontal segments and 4 is the number of vertical segments.



The number of crosses reveals the product: in this case, 8. This diagram can also be rotated 90°, to show 4 × 2. The cross-product technique is also limited in scope since it becomes inefficient to count crosses or intersections when the result exceeds 24.

The set of basic multiplication number facts generates a host of issues for teachers. One is, which symbol is best for representing the multiplication operation? The most common symbol has traditionally been ‘x’. However, with algebra topics appearing earlier in the mathematics curriculum (in the primary grades rather than after eighth grade), the x symbol can cause confusion. The symbol looks like *x*, the variable symbol. Some schools and teachers therefore use alternate ways to represent multiplication, such as (2)(4) or 2 • 4 or 2 * 4. The * symbol seems to be gaining in popularity and may take over in coming years.

Another issue concerns the scope of recall facts. Today, most mathematics programs require mastery through 9 x 9, or 81. There is some interest in returning to 12 x 12, or 144, which used to be the standard. Related to this concern is the evidence that certain multiplication number facts are more difficult for some learners than other facts. Usually the difficulty appears with combinations of 6s, 7s, 8s, and 9s. Increasing the recall set to include 10s, 11s, and 12s could compound learners’ difficulties with rapid recall. A few protocols in this section will include activities to practice the multiplication of 6s, 7s, 8s, and 9s.

Strategy for Presenting Multiplication Facts

One effective strategy for teaching the multiplication facts is to present them in specific stages. Below we provide five activities that, when done in sequence, build learners' confidence and ease the path to automaticity. The first four activities cover 75 of the 100 basic multiplication number facts to 81.

Stage 1: Doubles

Start teaching the basic multiplication facts with 2, using the concept of doubles. This stage easily relates to addition since it is equivalent to the doubles activity used when teaching addition. Learners can quickly tuck 20 multiplication facts under their belts.

	0	1	2	3	4	5	6	7	8	9
0			0							
1			2							
2	0	2	4	6	8	10	12	14	16	18
3			6							
4			8							
5			10							
6			12							
7			14							
8			16							
9			18							

Stage 2: Fives

Next, move on to the 5s. These multiplication number facts are easily recalled with only a little practice. One effective way for learners to practice the 5s to is to skip count to 45. This establishes confidence in their knowledge of multiplication facts. After this stage, they've learned another 20 multiplication facts.

	0	1	2	3	4	5	6	7	8	9
0			0			0				
1			2			5				
2	0	2	4	6	8	10	12	14	16	18
3			6			15				
4			8			20				
5	0	5	10	15	20	25	30	35	40	45
6			12			30				
7			14			35				
8			16			40				
9			18			45				

Stage 3: Ones and Zeros

Now present the 0 and 1 multiplication facts. Some learners may confuse the result of multiplying by zero with the result of adding zero. They may switch the results, to incorrectly recall 0 + 5 as having a sum of 0 and 0 × 5 as having a product of 5. The following matrix should assist in removing this confusion.

	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6			15				
4	0	4	8			20				
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12			30				
7	0	7	14			35				
8	0	8	16			40				
9	0	9	18			45				

Note: In order to show the pattern completely some of the basic multiplication number facts are included again in this stage, e.g., 5 × 0, 0 × 5.

Stage 4: Nines

The 9s offer an opportunity to explore an interesting pattern. Multiplying a single digit by 9 yields a result in which the sum of the two digits is 9. For example, 3 × 9 = 27 and 2 + 7 = 9. This pattern is valid for all values multiplied by 9 when you allow the recursion procedure. Consider 165 × 9. The product is 1485. Adding the digits 1 + 4 + 8 + 5 results in 18, and 1 + 8 equals 9. Here's another example: 12,985 × 9 = 116,865; 1 + 1 + 6 + 8 + 6 + 5 = 27; 2 + 7 = 9.

	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6			15				27
4	0	4	8			20				36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12			30				54
7	0	7	14			35				63
8	0	8	16			40				72
9	0	9	18	27	36	45	54	63	72	81

Note: In order to show the pattern completely some of the multiplication number facts are included again in this stage e.g. 1 × 9, 9 × 2.

Stage 5: Residues

The 25 remaining basic multiplication number facts are actually only 15, since there are 10 pairs of turnarounds (which demonstrate the commutative property: 3 × 4 and 4 × 3, 7 × 6, and 6 × 7, for example). But these 15 basic multiplication number facts include some of the hardest ones for learners to recall both rapidly and accurately. Therefore, learners may need more practice with these 15 facts to reach confidence. Only then can they begin the journey to automaticity of recall for these facts.

	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

A journey starting with the concept of multiplication, moving on through practice to establish confidence, and finally reaching automaticity with basic multiplication facts can seem tedious and may be bumpy. But once automaticity has been achieved, learners are much more likely to succeed in advanced mathematics.

Division

For some learners, recalling basic division number facts seems to be more difficult than recalling multiplication facts. This may be due to the natural tendency to view a division number fact as a "special" case of a multiplication number fact. For example, 48 ÷ 6 is often viewed as an unknown number times 6 that results in 48, or 6 × [ ] = 48. This extra derivation step (recalling the multiplication fact) can result in a slower recall rate for division facts. While a solid recall rate for multiplication facts should result in a reasonable recall rate for division, if students have not achieved automaticity with multiplication, they will stumble over division. Therefore, practice with division number facts should occur only after the learner has a reached automaticity of the multiplication facts, or has at least obtained confidence with focused practice.

Sometimes learners are confused by the symbols used to represent the operation of division. For example, students may misrepresent 18 ÷ 6 when asked to write it in the long division format. They may write $18\overline{)\,6\,}$, mimicking the left to right order of the original format. This may only be an error of format, however, and not of misunderstanding the concept of division.

Because of the relationships among the four arithmetic operations, teachers can introduce and develop the concept of division from multiplication and subtraction constructs. One common approach to the concept of division is sharing, or partitioning. Another approach is grouping, or repeated subtraction.

Sharing

The idea of sharing is established early through children's experiences ("Charlie, share your cookies with your friends!" "Juanita, be nice and share your toys!"), so they come to school with a definite understanding of the concept of division. Of all the mathematics operations, then, the concept of division may be the easiest for learners to grasp. When introducing division through the sharing approach—an approach that asks the child to determine how to share a collection of objects into a given number of equal parts—it may be useful to tap the child's experience. For example, we can present a problem like this: a child has 8 toys and there are four children playing with the toys. What number of toys should each child have to lessen the tension and give each child a fair share?

The division statement is $8 \div 4$ or $8 = 4 \times [\ \ ]$. This method presents division in terms of the missing factor in a multiplication statement, reinforcing the relationship between division and multiplication. And because of this connection, most of the activities for developing the concept of multiplication can be used to develop the concept of division.

Grouping

The grouping approach to teaching division is appropriate when learners are confronted with a collection of objects and asked to make groups of a given number. Grouping can also be described as repeated subtraction. In this method, learners create smaller groups from the initial group. So, if there are 24 chairs in a class and we want to form small groups of 6 chairs each, how many groups can we form? This problem can be stated as $24 \div 6$, or as $24 - 6 = 18$, $18 - 6 = 12$, $12 - 6 = 6$, $6 - 6 = 0$. In this subtraction sequence, there are four repeated subtractions, so $24 \div 6 = 4$.

In terms of helping learners gain confidence in their knowledge of basic division *facts*, the sharing approach is more effective than the grouping approach. The grouping approach is more helpful in developing the *concept* of division.

Zero

The interplay between zero and division is interesting. The simple approach to this topic is to tell learners that you cannot divide by zero. However, there is more to understanding division by zero, and giving learners this background will enhance their understanding of division. Zero can occur in three positions in the binary operation of division. It could be the numerator only, the denominator only, or both numerator and denominator.

For example, the three possible statements could be $0 \div 3 = [\ ]$, $3 \div 0 = [\ ]$, and $0 \div 0 = [\ ]$. Using the sharing approach, each of these division statements can be written as a missing factor multiplication statement. Thus, $0 \div 3$ becomes $0 = 3 \times [\ ]$. The only result that would satisfy this statement is 0. Rewriting $3 \div 0$ as a multiplication statement yields $3 = 0 \times [\ ]$. Within our number system there is no value that satisfies this statement. A coherent mathematical system avoids the possibility of no possible value answers, so this operation is not allowed. Finally, $0 \div 0$ can be rewritten as $0 = 0 \times [\ ]$. In our number system, all possible numbers satisfy this multiplication statement. Again, a coherent mathematical system does not "like" it when every number is valid, so it is not allowed. For these two reasons, then, division by zero is not allowed.

Informal Diagnostic Strategies

Since many mathematical algorithms are built on a solid recall of multiplication number facts, incorrect recall of these basic facts can thwart mathematical advancement. A slight hesitancy or lack of confidence in recalling a group of basic multiplication number facts can affect a learner's mastery of many different topics in the intermediate and middle school mathematics curriculum. Detecting errors early is therefore crucial. A diagnostic cycle hopefully can correct misunderstandings before they lead to more serious situations. This diagnostic cycle starts with the **detection** of an error pattern in recalling a multiplication (or any other arithmetic operation) number fact. The next stage, **diagnosis**, is when the teacher figures out what is happening—where and why the learner makes the mistake. This stage requires teachers to interview the learner, asking specific probing questions to clearly determine what the difficulty or misunderstanding is. An individual interview could be as short as ten minutes. The teacher would present the missed number facts and ask the learner to demonstrate the approach taken to recall the result. *Listening* to the learner is key at this point—instruction can wait until the next stage of the diagnostic cycle, **prescription**. After determining what the mistake is, teachers prescribe corrective action. The learner practices the suggested action and then the results are assessed and monitored. From these **assessments** an **evaluation** is made of the correction of the misunderstanding. If the misunderstanding still exists, the cycle is repeated with new data.

Since many mathematical algorithms are built on a solid recall of multiplication number facts, incorrect recall of these basic facts can thwart mathematical advancement.

This diagnostic paradigm should trigger a variety of interventions to help learners with the basic multiplication and division number facts.

Strategies for Addition and Subtraction Number Facts

Section 1, *Early Numerical Skills (Grades 1–3)*, describes a variety of intervention approaches to help learners acquire rapid recall of addition and subtraction number facts. Students will most likely need extra practice with the 7s, 8s, and 9s, typical problem areas. Some of the protocols found in Section 1 offer practice with these addition and subtraction facts. The goal for recall with automaticity is 2 or 3 seconds for addition facts up to 18. Students should be able to answer at this rate by late grade 2. For subtraction, the same recall rate should be reached in late grade 3.

Strategies for Multiplication and Division Number Facts

As mentioned earlier, the focus for strategies should be on the multiplication number facts to 81, since division facts are usually viewed as missing factor multiplication statements. The strategies appropriate for the operation of multiplication should be easily adapted to apply to the operation of division.

A possible strategy would be to use the multiplication matrix to search for patterns.This activity would also be an excellent review of which multiplication number facts learners know, which ones they're shaky on, and which ones they do not know.

	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

In this method, learners cross off the multiplication number facts in a systematic manner. A possible sequence is as follows.

- First, ask learners to find and cross off all the products of
  - 0 and the numbers 0–9
  - 1 and the numbers 0–9
  - 2 and the numbers 0–9
  - 5 and the numbers 0–9
  - 9 and the numbers 0–9
- Next, ask learners to find and cross off all the "double" products (square numbers), that is, 1 × 1, 2 × 2, 3 × 3, etc. There will only be five of these doubles not already crossed off : 3 × 3, 4 × 4, 6 × 6, 7 × 7, and 8 × 8.
- The remaining values not crossed off are what may be called the "Ten Favorites."
  - The combinations are

    3 × 4 3 × 6 3 × 7 3 × 8

    4 × 6 4 × 7 4 × 8

    6 × 7 6 × 8

    7 × 8

    and their turnarounds. Students should practice the Ten Favorites to reach confidence with them before attempting automaticity. Sometimes 6 × 9, 7 × 9, and 8 × 9 are also included in the set of multiplication number facts that require extra practice.

Flashcards are a time-tested technique for building multiplication and division fact mastery. They're useful because they're fun, motivating learners to learn their facts. But flashcards are most effective if used strategically. There are 100 basic multiplication number facts. Viewing all 100 flashcards in every practice session has limited value. A better approach is to remove all but 10 of the known number facts, leaving these to be mixed with some of the 15 unknown number facts. As the learner practices, he or she could separate the flashcards into two piles, one labeled **Got It** and the other labeled **Close**. The goal is to have all the flashcards on the **Got It** pile after a session of practicing to gain confidence.

It is typical for learners to reach efficient recall of basic division number facts many months after they've mastered the basic multiplication number facts. The following chart shows reasonable expectations for multiplication and division number fact concept development, practice recall at a rate of 3 or 4 seconds, and automaticity recall at a rate of 2 or 3 seconds. These are general expectations that will not align with all students—ELL, special needs, and gifted learners' rates will differ. However, these benchmarks offer reasonable parameters.

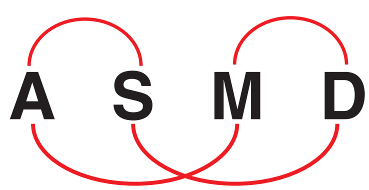
Time	**Concept Development**	**Practice Recall Rate: 3 or 4 Seconds**	**Automaticity Recall Rate: 2 or 3 Seconds**
late grade 3	multiplication to 81		
early grade 4	division from 81		
mid grade 4		multiplication to 81	
late grade 4		division from 81	multiplication to 81
late grade 5			division from 81

Within these grade levels a variety of activities and techniques can be designed to practice with confidence the basic multiplication and division number facts. The protocols in the next section are designed to develop learners' confidence, and with repeated practice, lead to automaticity of the basic multiplication and division number facts.

Tips for Success

Here are some helpful reminders for teaching recall of the basic number facts.

- Keep the mode sequence as the keystone principle of instruction: hands-on to pictorial to abstract.
- Develop the concept, then have students practice to gain confidence, and next introduce the goal of automaticity.
- Build the process from the interrelationships of addition, subtraction, multiplication, and division.



- Help students succeed by mixing together easy and hard number facts during early practice sessions.
- Design engaging activities that motivate the learner to practice the basic multiplication number facts.

- Practice the basic multiplication number facts using scenarios from daily life.
- Use a multiplication matrix to visually reinforce the commutative property and other specific properties of the basic multiplication number facts before moving to rapid recall.
- Learning the multiplication number facts takes time; the most efficient teaching approach combines brief periodic interventions and regular practice.
- Follow a reasonable sequence of learning the multiplication number facts. Do not go in order from 0 to 9.
- Remember that mathematics is the science and language of patterns. Look for patterns!

Protocols

A successful numeracy development program requires monitoring of learners' progress as they practice, with interventions between the practices. We have provided some intervention activities below. The protocols are not meant to be full-fledged lesson plans, but rather basic frameworks that can be used by classroom teachers, parents, guardians or teacher's aides. They include a specific topic, pre-assessment information, appropriate age level, sequence of actions, an assessment strategy for evaluation, and suggestions for adaptations or modifications for English Language Learners (ELL) and special needs learners and gifted learners.

Protocol Matrix

Title	**Topic**	**Level**	**Instructional Mode**
Up and Down by One	addition number patterns, complementary addend changes	3rd grade and up	small group (pairs)
More	addition with one-more-than and two-more-than	3rd grade and up	small group, individual
Less	subtraction with one-less-than and two-less-than	3rd grade and up	small group, individual
Multiplication Intersections	beginning multiplication	3rd grade and up	small group, individual
Cover Over	relationship of repeated addition to multiplication	4th grade and up	small group, individual
Stuck on Your Head	reinforce multiplication and addition number facts	4th grade and up	small group, individual

Protocol Matrix			
Title	**Topic**	**Level**	**Instructional Mode**
Divide Up	concept of division related to repeated subtraction	3rd grade or early 4th grade and up	small group, individual
Factor Rectangles	relating multiplication to arrays or rectangles	4th grade and up	small group, individual
Wonderful Nines	multiplication number patterns for the basic facts of 9	5th grade and up	small group,
Messy Multiplication	multiplication patterns	5th grade and up	small group, individual
Close Enough?	division near facts, missing factor multiplication	5th grade and up	small group, individual
Messy Division	division patterns	6th grade and up	small group, individual
Hit the Target Num-ber	arithmetic operations practice (+, -, x, ÷)	6th grade and up	small group (3 or 4 learners)
Multiplication Balance	multiplication number facts to 81	3rd grade and up	small group, individual
Division Balance	division number facts from 81	4th grade and up	small group, individual

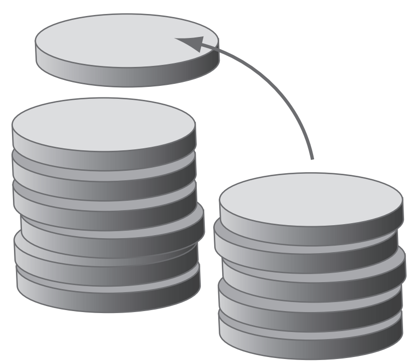
Up and Down by One

Topic: Addition number patterns, complementary addend changes **Level:** 3rd grade and up
Instructional Mode: Small group (pairs) **Time:** 20–30 minutes

Pre-assessment
Check understanding of two groups or numbers representing groups that can be added together to give a result or sum.

Materials
- 30 counters
- 20 or so statement cards that show addition combinations (e.g., 6 + 5, 7 + 7, 9 + 3)

Launch
This activity moves counters from one group to another to show how different sets of addends can result in the same sum. Demonstrate how it works:
- Create two sets of 6 counters.
- Ask learners what the addition statement for this grouping is. (6 + 6) Ask them what the sum is. (12)
- Take 1 counter from one set and move it to the other set. This is the "up-and-down-one" action.
- Ask learners what the addition statement is now. (5 + 7) Point out that both situations have the same sum. (12)



Explore
- In pairs, have one learner choose a statement card and read it aloud to the other learner.
- The second learner will use counters to show the statement, and then write the statement and sum.
- The second learner then moves one counter in the "up-and-down-one" action, and writes this new addition statement and sum.

Students switch roles and repeat the activity.

Closure
Give the statement 9 + 7 and have the learners show up-and-down-one using the counters.

Assessment	Evaluation
Have the learner show the sum of the statement card using counters.	Success will be an accurate physical representation.
Now have the learner do the up-one-down-one action, and write the new addition equation for the physical representation.	Success will be an accurate addition equation.
Ask learners to try to answer without counting each set to get the result.	Success will be stating the sum without counting the groups to find the total.

Adaptations for Various Learners

ELL:

- Demonstrate the action using diagrams.
- Explain with counters the meaning of up and down by one.

Special Needs:

- Demonstrate the action using diagrams.
- Explain with counters the meaning of up and down by one.
- Only use values that result in single-digit sums in the beginning. Later include two-digit sums.

Gifted:

- Have learners explain what is happening if you go up and down by 2. For example, starting with groups of 8 and 7, moving 2 counters yields 6 and 9. Does the activity work for a change of 2?
- Have learners explain other-up-and-down values.
- As a final challenge, have learners show the values as an equation. For example, 8 + 9, up and down by 3, is 5 + 12 or (8 – 3) + (9 + 3) = 17.

More

Topic: Addition with one-more-than and two-more-than
Instructional Mode: Small group, individual
Level: 3rd grade and up
Time: 20–30 minutes

Pre-assessment
Check understanding of the meaning of one-more-than and two-more-than.

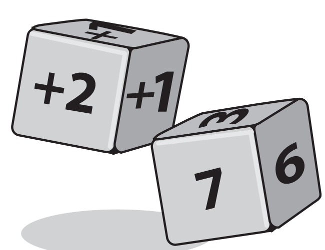
Materials
- A number cube with the faces marked +1, +1, + 2, and +2 (one more and two more)
- A number cube with the faces marked 3, 4, 5, 6, 7, and 8 or with any set of values needing practice

Launch
This activity involves orally expressing a value plus one more or plus two more.

Demonstrate the activity:
- Roll both number cubes.
- Have a learner say the equation shown by the cubes.
  For example, if one number cube shows +1 and the other shows 7, the learner would say, "seven and one more is eight" or "seven plus one equals eight."

Explore
- Learners can do this activity individually or in pairs, with one rolling and the other saying the equation. Each learner will state the resulting equation from the two number cubes.
- Learners should continue until they have encountered most of the 24 possible combinations. Show learners how to make a simple chart to keep track.

Closure
- Roll both number cubes.
- Reinforce the idea of one more or two more rather than counting on from the larger value. It is more efficient to see 7 and 2 as two more than 7, or 9, instead of counting up from 7 to 8 to 9.



Assessment	Evaluation
Roll both number cubes.	Success is stating accurately the one-more and two-more expressions of addition.
Have the learner state aloud the basic addition number fact shown on the cubes.	
Continue until both one-more and two-more situations are expressed.	

Adaptations for Various Learners

ELL:

- Demonstrate both a one-more and a two-more roll of the number cubes.
- Reinforce viewing the expression as one-more or two-more rather than counting up.

Special Needs:

- Demonstrate both a one-more and a two-more roll of the number cubes. Reinforce viewing the expression as one-more or two-more rather than counting up.
- Change the values on the one number cube to 3, 3, 4, 4, 5, and 5.
- Later replace the number cube face values with 3 through 8.

Gifted:

- Change the value on the one number cube to 7, 7, 8, 8, 9, and 9.
- Later change the values on the other number cube to +1, +1, +2, +2, +3, and +3.
- Ask learners to write the expression as well as say the number fact aloud.

Less

Topic: Subtraction with one-less-than and two-less-than
Instructional Mode: Small group, individual
Level: 3rd grade and up
Time: 20–30 minutes

Pre-assessment
Check understanding of the meaning of one-less-than and two-less-than.

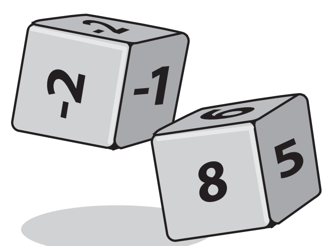
Materials
- A number cube with the faces marked –1, –1, –2, and –2 (one less and two less)
- A number cube with the faces marked 4, 5, 6, 7, 8, and 9 or any set of values needing practice

Launch
This activity involves orally expressing a value one less or two less.

Demonstrate:
- Roll both number cubes.
  Example: the one number cube shows –2 and the other number cube shows 6.
- Have a learner say the equation shown on the cubes: in this case, “six and two less is four,” or “six minus two is four.”

Explore
- Either individually or in pairs, have the learners roll both number cubes.
- Each learner will state the resulting subtraction equation from the two number cubes.
- Continue until most of the 24 possible combinations are expressed. Show learners how to make a simple chart to keep track.

Closure
- Roll both number cubes.
- Reinforce the idea of one less or two less rather than counting down from the larger value. It is more efficient to see 8 and –2 as two less than 8, or 6, instead of counting down from 8 to 7 to 6.



Assessment	Evaluation
Learners roll both number cubes.	Success is stating accurately the one less and two less expressions of subtraction.
Have the learner state orally the basic subtraction number fact.	
Continue until both one less and two less situations are expressed.	

Adaptations for Various Learners

ELL:

- Demonstrate both a one-less and a two-less roll of the number cubes.
- Reinforce viewing the expression as one less or two less rather than counting down.

Special Needs:

- Demonstrate both a one-less and a two-less roll of the number cubes.
- Reinforce viewing the expression as one less or two less rather than counting down.
- Change the values on the one number cube to 3, 3, 4, 4, 5, and 5.
- Later replace the number cube with face values 3 through 8.

Gifted:

- Change the value on the one number cube to 7, 7, 8, 8, 9, and 9.
- Later, change the values on the other number cube to –1, –1, –2, –2, –3, and –3.
- Have the learner write the expression as well as state the basic subtraction number fact.

Multiplication Intersections

Topic: Beginning multiplication (products less than 25) **Level:** 3rd grade and up
Instructional Mode: Small group (2 learners), individual **Time:** 20–30 minutes

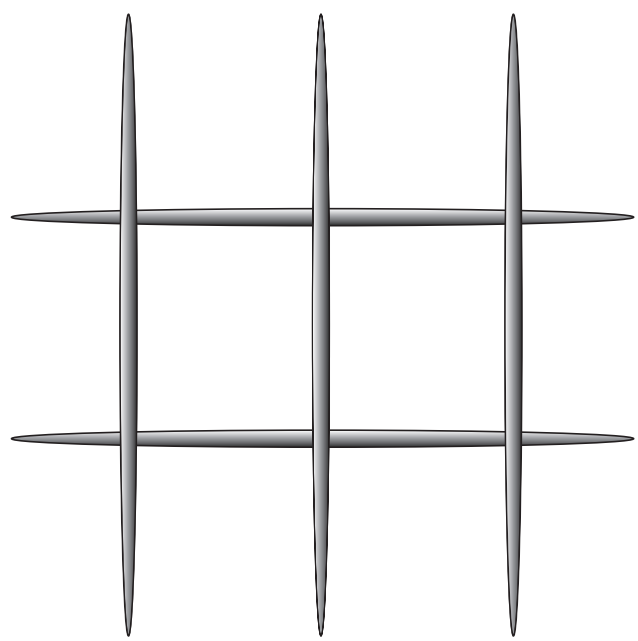
Pre-assessment
Check for understanding of beginning multiplication number facts. Check for understanding of the words *horizontal, vertical, street, avenue,* and *intersection.*

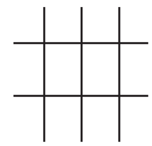
Materials
- 25 toothpicks or wooden sticks per small group
- Recording chart, as below

Number of Avenues (horizontal)	**Number of Streets** (vertical)	**Number of Intersections**	**Map of Town**	**Multiplication Fact**

Launch
In this activity, learners will build the roads in a town as a hands-on way to understand multiplication facts.

Demonstrate how to construct the roads in our town. Horizontal roads will be called avenues and vertical roads will be called streets. Where a street and an avenue cross is an intersection. Each toothpick or stick is one road.
- Make two avenues and three streets.
- Locate each intersection, and count them.
- Record the information in the chart.



Number of Avenues (horizontal)	**Number of Streets** (vertical)	**Number of Intersections**	**Map of Town**	**Multiplication Fact**
2	3	6		2 x 3 = 6

Explore

- Each pair of learners needs 25 wooden sticks and a recording sheet.
- The learners decide how to build their roads in their town. They must start with 2 horizontal roads (avenues) and build the vertical roads (streets) one at a time and record the information in the chart.
- When finished with the 2-avenue town they can begin new towns with 3, 4, and 5 avenues.
- Continue until the 2s, 3s, 4s, and 5s have been recorded.

Closure

Ask a learner to explain the number of intersections in a town with 5 avenues and 4 streets.

Assessment	***Evaluation***
Ask learners to start with a 4-avenue town.	
Now ask them to add 3 streets.	
How many intersections are in this town?	Success is verified by a response of 12 intersections.
Ask learners to state the multiplication number fact for this town.	Success is verified by a response of 4 x 3 = 12.

Adaptations for Various Learners

ELL:

- Demonstrate the situation of avenues and streets by words, diagrams, and multiplication statements.
- Begin with 16 wooden sticks.
- Later use more wooden sticks.

Special Needs:

- Demonstrate the situation of avenues and streets by words, diagrams, and multiplication statements.
- Begin with 10 wooden sticks.
- Later involve more wooden sticks, up to 16.

Gifted:

- Learners should look for patterns in the number of avenues, streets, and intersections.
- Learners could explore larger towns (more avenues and streets) without the wooden sticks. The situations could be shown by either diagrams or multiplication statements.

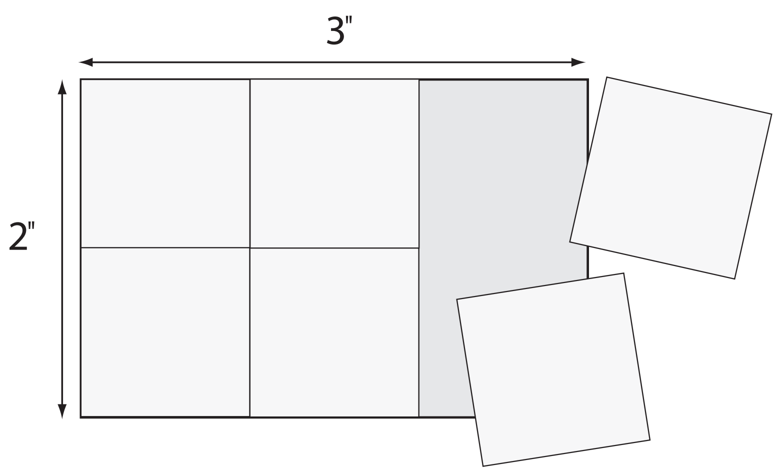
Cover Over

Topic: Relationship of repeated addition to multiplication
Level: 4th grade and up
Instructional Mode: Small group, individual
Time: 20–30 minutes

Pre-assessment
Check understanding of repeated addition and the skill of covering a rectangular area with square tiles.

Materials
- 1" square tiles (paper or other material)
- 6" x 8" (or larger) rectangular outline, without grid lines, on a sheet of paper

Launch
- Demonstrate how to cover a small rectangle (2" x 3") with 1" squares.
- Demonstrate that the tiles can also be formed into rectangles, for example 1" × 3" and 1" × 3" rectangles, or 1" × 2", 1" × 2", and 1" × 2" rectangles, to cover the 2" x 3" rectangle.



Explore
- Give each small group one rectangular outline (you may use various size rectangles). The goal of the activity is to determine the number of 1" tiles needed to cover the rectangles.
- First, learners should estimate the number of 1" tiles needed to cover the rectangle.
- Second, they use the tiles to verify the estimate.
- Third, they write the multiplication expression for covering the rectangle.
- Fourth, learners look for other possible rectangles that would cover the given rectangle. Ask them if they can find several combinations.

Closure

Ask learners to explain the relationships among multiplication, repeated addition, and the "cover over" area. This may be best done with a specific example.

Assessment	***Evaluation***
Give learners a rectangle, 3" x 4". Ask them to:	
Estimate the number of tiles that would cover this rectangle.	—
Determine the number of tiles that would cover this rectangle.	Success would be 12 tiles.
Write the related multiplication number fact.	Success would be 3 × 4 = 12.
Write the repeated addition expression for this multiplication number fact.	Success would be 3 + 3 + 3 + 3 or 4 + 4 + 4.

Adaptations for Various Learners

ELL:

- Show how to cover a rectangle with tiles.
- Encourage learners to draw diagrams or verbally explain the multiplication number fact and repeated addition expression.

Special Needs:

- Show how to cover a rectangle with tiles.
- Encourage learners to draw diagrams or verbally explain the multiplication number fact and repeated addition expression.
- Restrict the rectangles to an area less than 20" square.

Gifted:

- Expand the outline rectangles up to 12" × 12".
- Ask learners to find other rectangle shapes in the classroom and cover these shapes with 1" square tiles.
- Ask learners to explain how they determined the cover for complex rectangular shapes.

Stuck on Your Head

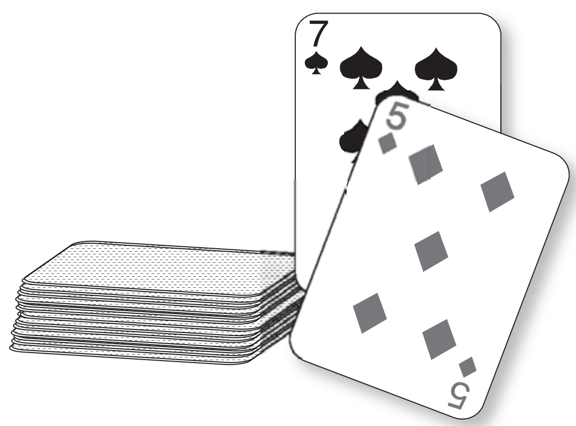
Topic: Reinforce multiplication or addition number facts | **Level:** 4th grade and up
Instructional Mode: Small groups of 3 learners | **Time:** 20–30 minutes

Pre-assessment
Check understanding of multiplication and addition concepts.

Materials
- Deck of cards with the four 10 cards and all the face cards removed

  or
- Number cards made from index cards, four each of the numbers 1 through 9

Launch
Explain that three learners play this game. The game starts with the deck of cards face down.
- The group chooses one person to be "it"—the person who will know the answer.
- The other two players will each draw a card from the deck, and without looking at its value, place the card on their foreheads with the number showing.
- The first learner (the "it" person) declares the product of the two numbers on the foreheads.
- The other players take turns guessing what number must be on his or her own forehead. The first one to correctly say his or her number gets the two cards ("wins").
- This person becomes "it," and the other two draw cards. Repeat steps.



Explore
- Each small group must have three learners. A balance of learners based on their ability to recall the multiplication number facts may be the most effective way to construct the small group.
- Following the rules of the game, these three learners take turns declaring the product shown on the foreheads.
- The first to correctly tell the number on his or her own forehead receives both cards and is "it."
- Continue to do this activity until all the cards are used.
- A "winner" can be declared by counting the number of cards in each player's stack.
- This game can be made less competitive if the learners with the cards on their foreheads write down their guess at the number on their forehead (within 4 seconds). A correct answer would be worth one point. Continue until all the cards have been used.

Closure

- Have each small group count up their scores.
- Have each small group report on what multiplication number facts seemed to be most difficult.
- Have each small group take those cards out and play a few rounds with those “tough” numbers.

Assessment	*Evaluation*
Ask a learner to name the missing factor, given one factor, and a product (e.g., 8 and 56).	Success is a correct response delivered verbally within 4 seconds.

Adaptations for Various Learners

ELL:

- Demonstrate the game by playing it and showing each of the rules and decisions needed by actions.
- Practice a few times to verify understanding.

Special Needs:

- Demonstrate the game by playing it and showing each of the rules and decisions needed by actions.
- Practice a few times to verify understanding.
- Change the game from multiplication number facts to addition number facts. Now the two numbers on the foreheads are added. The learner must tell one of the addends.
- Later introduce the multiplication number facts with only the values 1 through 6 in the number deck.
- Much later introduce the multiplication numbers facts using all values 1 through 9 in the number deck.

Gifted:

- Remove all the 1s, 2s, 3s, and 4s.
- Play the game with the restriction that the product and unknown factor must be stated within 3 seconds.
- Include more number cards: four cards each of 10s, 11s, and 12s and insert the 1s, 2s, 3s, and 4s number cards again.

Divide Up

Topic: Concept of division related to repeated subtraction
Level: 3rd or early 4th grade
Instructional Mode: Small group, individual
Time: 20–30 minutes

Pre-assessment

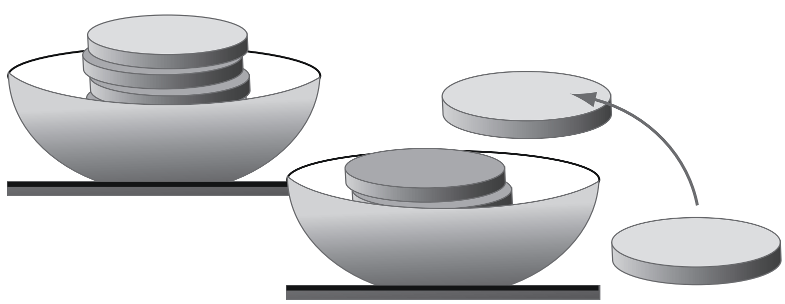
Check understanding of multiplication and the concept of repeated subtraction.

Materials

- 30 counters
- Containers or small cups that hold at least 6 counters each
- 20 or so statement cards that say, "How many sets of _____ counters can be made with _____ counters?" Replace the blanks with different numbers when you make the cards.
  (The statement cards will represent math facts, i.e., no remainders.)

Launch

- Create a group of 8 counters. Ask learners how many sets of 4 counters can be made from this group?
- Demonstrate counting out groups of 4: Put 4 counters in one container; then put another 4 counters in a second container. 8 – 4 = 4, 4 – 4 = 0. There are two sets of 4 counters, so 8 ÷ 4 = 2.



Explore

- Learners select a statement card and then, using counters and containers, show the number of sets needed to satisfy the conditions.
- Next, learners write the equation as a division operation, as a multiplication operation, and as repeated subtraction.
  For example, given this statement, "How many sets of 5 counters can be made with 15 counters?" learners would create 3 sets of 5 in the containers. Then they would write: 15 ÷ 5 = 3, 3 x 5 = 15, and 15 – 5 = 10, 10 – 5 = 5, 5 – 5 = 0.
- Learners should complete all 20 statement cards and their related equations.

Closure
Select one of the statement cards and have a learner construct the physical representation with containers and counters.

Assessment	*Evaluation*
Give learners a division statement, e.g., 24 ÷ 6, and ask them to construct the physical representation with counters and containers.	Success is showing accurately the physical representation.
Also have the learner write the related multiplication number fact (4 x 6 = 24) and the repeated subtraction operation (24 – 6 = 18, 18 – 6 = 12, 12 – 6 = 6, 6 – 6 = 0).	Success is also showing the related multiplication operation. The repeated subtraction relationship is important but the crucial understanding is the relationship of multiplication and division.

Adaptations for Various Learners
ELL:
- Demonstrate the action of putting counters into containers to show division.
- Use diagrams to show the physical representation and its relationship to multiplication and repeated subtraction.

Special Needs:
- Demonstrate the action of putting counters into containers to show division.
- Use diagrams to show the physical representation and its relationship to multiplication and repeated subtraction.
- Begin with single digits, for example, 9 ÷ 3 or 8 ÷ 4.
- Later use double digits, for example, 16 ÷ 8 and 18 ÷ 3.

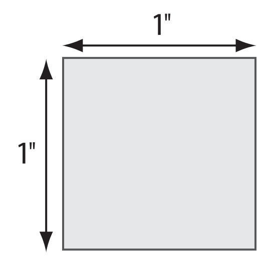
Gifted:
- Make statement cards that include more advanced division statements such as 56 ÷ 8 and 63 ÷ 7.
- Include a few statement cards that result in a remainder, for example, 22 ÷ 7 or 33 ÷ 6. Ask learners to write the related multiplication operation with remainders. So, 22 ÷ 7 would be written 3 x 7 + 1 = 22.

Factor Rectangles

Topic: Relating multiplication to arrays or rectangles
Instructional Mode: Small group, individual
Level: 4th grade and up
Time: 20–30 minutes

Pre-assessment
Check understanding of the basic multiplication number facts. Also check that learners understand that two factors are needed for a multiplication operation.

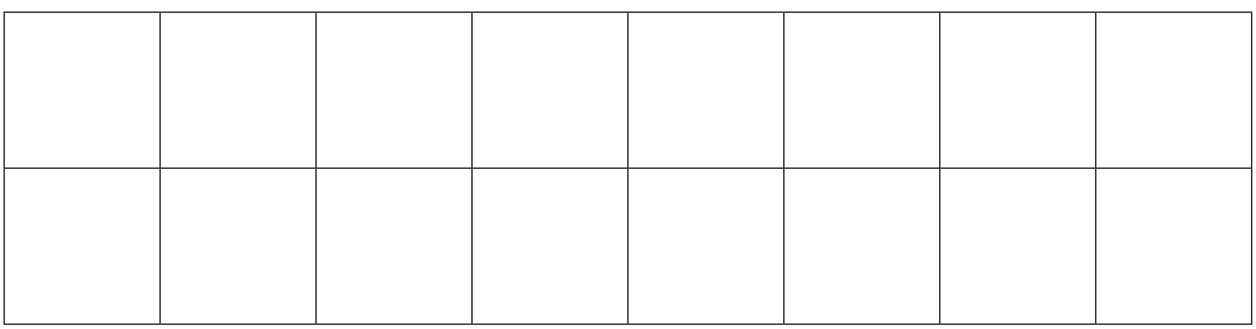
Materials



- Grid with squares (at least 20 rows by 20 columns, see Appendix, page 117)
  or
- Square tiles (1" tiles, see Appendix, page 118)
- Number cards with values from 8 through 81 (products of basic multiplication math facts)

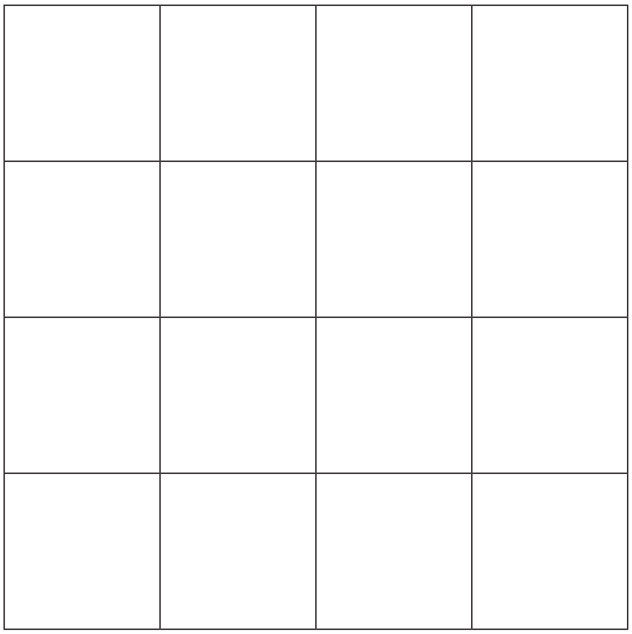
Launch
In this activity, learners will find all the factors of a number using either the grid or the tiles.

- Demonstrate the activity by choosing a value and showing the results, as follows.
- For example, for the value 16, create a rectangle of 16 squares on the grid (by drawing an outline) or using the tiles. You might first create one like this:

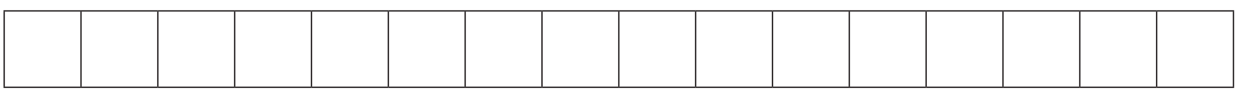


- Then explain that there are 2 rows of 8 squares, or 8 x 2 = 16. So, 8 and 2 are factors of 16. You might then ask learners, “Does an 8 by 2 rectangle have the same area as a 2 by 8 rectangle?”

- Next, outline a 4 x 4 rectangle on the grid or make one with tiles. (Remind students that a square is a rectangle, just a special type that has its own name.)



- And finally, make a 1 x 16 rectangle.



- Now list the factors of 16: 1, 2, 4, 8, and 16.

Explore

Here's how learners proceed.

- First, they'll choose a number card.
- Then, they'll find the factors by building rectangles from the grid or the tiles. (The tiles actually are more efficient.)
- When each rectangle is constructed, learners should write down the multiplication number fact.
- When learners have found all the possible rectangles that can be formed from the chosen value, they list all of its factors.
- Learners should create rectangles for at least 15 number cards.

Closure

- Choose a value with several factors, for example, 36.
- Ask learners to explain how the process of forming rectangles is completed.

Assessment	*Evaluation*
Using the chosen value in the closure section, ask learners to form the appropriate rectangles.	Success is accurate rectangles representing the chosen value.
Ask learners to write the multiplication basic facts for each rectangle.	Success is appropriate multiplication number facts related to the chosen value.
Ask learners to write the factors of the given value.	Success is listing *all* the factors of the chosen value.

Adaptations for Various Learners

ELL:

- Demonstrate the action of forming rectangles using more than one example.
- Write one sample multiplication number fact that is shown by the grid or tiles.

Special Needs:

- Demonstrate the action of forming rectangles using more than one example.
- Write one sample multiplication number fact that is shown by the grid or tiles.
- Use number cards showing only the values of 24 or less.

Gifted:

- Ask learners to find the numbers for which they can only build two unique rectangles or arrays (for example, 9. You can only build 3 x 3 and 9 x 1 rectangles). What are these numbers?
- Have learners look for patterns in the number of factors and the types of rectangles.

Wonderful Nines

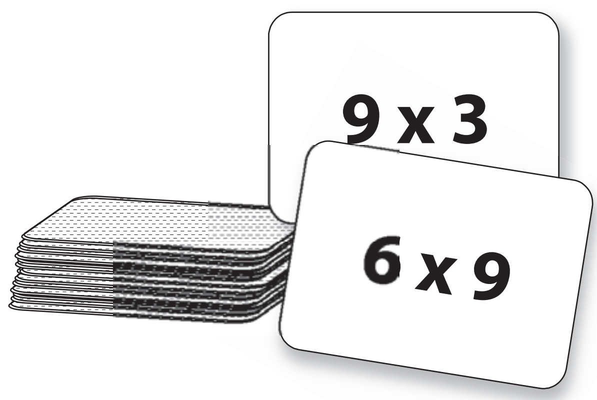
Topic: Multiplication number pattern for the basic facts of 9
Instructional Mode: Small group, individual
Level: 5th grade and up
Time: 20–30 minutes

Pre-assessment
Check understanding of the concept of multiplication. Check understanding of place value, especially the tens place.

Materials
- Multiplication statement cards with all possible combinations of 9 and the values 2 through 9—in other words, 2 x 9, 9 x 2, 3 x 9, 9 x 3, etc. —15 cards total
- 9 Multiplication—Fact Matrix (see Appendix, page 119; optional for use with gifted, accelerated learners)

Launch
This activity offers a handy method for remembering the 9 multiplication facts.
Explain the pattern for the Wonderful Nines:
- The tens place of the result is always 1 less than the non-9 factor.
- The sum of the two digits of the result is always 9.

  So, for example, to remember the product of 6 × 9, take 1 less than 6, which is 5; 5 and 4 make 9; so the product is 54.



Explore
- Either individually or in pairs, ask learners to choose a statement card and practice the Wonderful Nines pattern.
- Continue until all 15 statement cards have been used.
- Shuffle the statement cards and repeat the activity.

Closure
- Choose one of the statement cards.
- Have learners explain the process of obtaining the multiplication number fact using the Wonderful Nines technique.

Assessment	Evaluation
Ask learners to choose a statement card.	Success is applying the Wonderful Nines number pattern technique resulting in a correct product.
Have a learner apply the Wonderful Nines number pattern technique to obtain the product.	

Adaptations for Various Learners

ELL:

- Provide learners with the 9 multiplication-fact matrix (see below).
- Write out the step-by-step process of the Wonderful Nines number pattern technique.

Special Needs:

- Provide learners with a 9 multiplication-fact matrix (see below).
- Write out the step-by-step process of the Wonderful Nines number pattern technique.
- Begin with 9 multiplied by 2s, 3s, and 4s.
- Later include 5s, 6s, 7s, 8s, and 9s multiplied by 9.

Gifted:

- Have learners construct a new 9 multiplication-fact matrix with the Wonderful Nines number pattern technique shown for each cell in the matrix up to 9 × 9. See Appendix.
- Have learners explore other 9 number patterns, for example, 9 × 12, 9 × 11, 9 × 10.

	0	1	2	3	4	5	6	7	8	9
0										
1										
2										18
3										27
4										36
5										45
6										54
7										63
8										72
9			18	27	36	45	54	63	72	81
10										
11										
12										

Messy Multiplication

Topic: Multiplication patterns
Instructional Mode: Small group (2 learners), individual
Level: 5th grade and up
Time: 20–30 minutes

Pre-assessment
Check understanding of basic multiplication facts. Check understanding of the words *factor, double, triple,* and *halve*.

Materials
- 16 statement cards with the following multiplication operations: 2 × 2, 2 × 3, 2 × 4, 2 × 5, 3 × 2, 3 × 3, 3 × 4, 3 × 5, 4 × 2, 4 × 3, 4 × 4, 4 × 5, 5 × 2, 5 × 3, 5 × 4, 5 × 5
- 5 word statement cards: *double one factor, double both factors, triple one factor, halve one factor,* and *double one factor and halve the other factor*
- Recording chart with following column headers and 16 rows below (see Appendix, page 120, for reproducible form)

Start Factors and Product	Word Statement Card	New Factors and Product	What Happened?

Launch
Demonstrate the activity using the 4 x 3 statement card.
- Ask a learner to give the product of the two factors. (12)
- Then ask what learners think would happen to the product if we doubled one factor.
- Check out their speculation with 4 x 3. If the factor 4 is doubled to 8, we have 8 × 3 = 24. So we see that the product is also doubled when one factor is doubled.

Start Factors and Product	Word Statement Card	New Factors and Product	What Happened?
4 × 3 = 12	double one factor	8 × 3 = 24	product doubled

- Explain that some combinations of multiplication statement card and word card will not work for this activity. An “inappropriate” word card would be one that results in a fraction.
  - Give an example, such as the following.

    Number card: 5 × 5 Word card: *halve one factor*

    In this case, halving 5 yields the statement 2.5 × 5, which is not a basic multiplication fact. This word card is rejected and another word card is selected.

Explore

This activity works best with two learners acting as a team. However, an individual learner could also complete the activity.

- Arrange the 16 number statement cards face down in an array.
- Make a stack face down of the word statement cards.
- A learner selects one card from the array and one card from the word stack.
- First, the learner must state the multiplication fact and its product.
- Second, the learner must decide if the word statement card is appropriate for the selected number card. If not, another is selected.
- Third, the learner must state the new multiplication fact and give its product.
- Fourth, the learner must state what happened to the start multiplication factors and product after the word statement card is applied. The other learner records all the information in the recording chart.
- Learners change roles and continue selecting number cards and word cards until all of the number cards have been selected. The word cards are replaced in the stack after each use.

Closure

Ask learners to explain what happened to the product when they chose the word statement *double one factor*.

Assessment	***Evaluation***
Ask learners to select a number card and read out the number fact and its product. Example: 3×5	
Learners select a word card. Ask, "What is the new multiplication fact?" Example: *triple one factor* 9×5	
Ask, "What was the original and what is the new product?"	Success is verified by $3 \times 5 = 15$ and $9 \times 5 = 45$.
Ask, "What happened to the product?"	Success is verified when learners state that the product is tripled when one factor is tripled.

Adaptations for Various Learners

ELL:

- Demonstrate the meaning of the word statements.
- Begin by using only three of the word statement cards: *double one factor, double both factors,* and *triple one factor.*
- Later include all five word statement cards.

Special Needs:

- Demonstrate the meaning of the word statements.
- Begin by using only two of the word statement cards: *double one factor* and *double both factors*.
- Later add *triple one factor* to the word statement cards.

Gifted:

- Expand the number of statement cards to include 6s, 7s, 8s, and 9s.
- Include another word statement card in the stack: *halve both factors*.

Close Enough?

Topic: Division near facts, missing factor multiplication — **Level:** 5th grade and up
Instructional Mode: Small group, individual — **Time:** 20–30 minutes

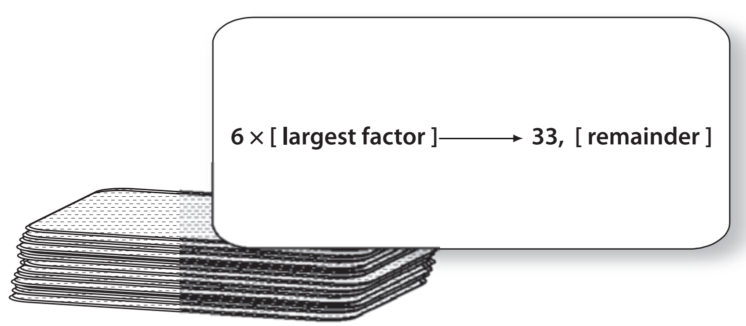
Pre-assessment
Check understanding of missing factors in multiplication situations. Check understanding of the meaning of remainder in division.

Materials
- 25 missing-factor statement cards with this format:
  4 x [largest factor] → 22, [remainder]
  3 x [largest factor] → 10, [remainder]
  Change the numbers on each card, making sure the “product” will result in a remainder.

Launch
This activity works best when learners team up in pairs, so that they can check the correct missing factor and remainder. If you plan to use it for individual learners, write the correct missing factor and remainder on the back of the statement card.

Demonstrate this activity with a selected statement card.
- Suppose the card says 6 x [largest factor] → 33, [remainder]
- Ask a learner to find the largest factor that, when multiplied by 6, will give a result as close to 33 as possible *without going over*.
- Ask the learner to declare what number is left. Explain that this is the remainder.
  So, in this case, the learner should declare that 5 is the largest factor and the remainder is 3.
- Practice at least two more statement cards with the learners to reinforce the process.

Explore
- Learners should practice until they have completed all 25 statement cards.
- When they have finished all 25, ask learners to select the five hardest statement cards; they can use these for additional practice.



Closure

Select one of the cards that learners chose as most difficult, and ask a volunteer to explain the process of finding the missing factor and remainder.

Assessment	***Evaluation***
Select two statement cards, one involving an 8 or 9 and another involving a 4 or 5.	Success is correctly answering both statement cards in 10 to 15 seconds.
Ask the learner to complete both cards.	

Adaptations for Various Learners

ELL:

- Explain the sequence of (mental) actions needed to complete both the missing factor and remainder.
- Define *largest factor* and *remainder* using specific examples.

Special Needs:

- Explain the sequence of (mental) actions needed to complete both the missing factor and remainder.
- Define *largest factor* and *remainder* using specific examples.
- Limit the statement cards to only those involving 1s, 2s, 3s, and 4s.
- Later include all the statement cards (25 statement cards).
- Allow learners to use a calculator for every other statement card.

Gifted:

- Include statement cards with 10s, 11s, and 12s as one of the factors.
- Modify the condition so that the closest factor *can go over* the target number. For example,
  7 × [largest factor] ⟶ 55, [remainder]
  Closest factor is 8 with -1 remainder.

Messy Division

Topic: Division patterns
Instructional Mode: Small group (pairs), individual
Level: 6th grade and up
Time: 20–30 minutes

Pre-assessment

Check understanding of basic division facts. Check understanding of the words *numerator, denominator, double,* and *halve*.

Materials

- 10 division statement cards as follows: 2 ÷ 2, 4 ÷ 2, 6 ÷ 2, 8 ÷ 2, 4 ÷ 4, 8 ÷ 4, 12 ÷ 4, 16 ÷ 4, 6 ÷ 6, 8 ÷ 8
- 5 word statement cards: *double the numerator, double the denominator, halve the numerator, halve the denominator,* and *halve the numerator and halve denominator*
- Recording chart with the following column headers and 10 blank rows (see Appendix, page 121, for reproducible form)

Start Division Fact and Result	Word Statement Card	New Division Fact and Result	What Happened?

Launch

This activity works best when two learners pair up as a team. However, an individual learner could also complete all the parts as well.

- Explain that this activity will help learners see patterns in division. Give one example. For instance, select the 8 ÷ 2 division statement card and the halve the numerator word card.
- Ask a learner to identify the numerator value and denominator value. (8 and 2, respectively)
- Ask the learner to state the result of 8 ÷ 2. (4)
- Now ask what would happen to the result if we halved the numerator? 8 ÷ 2 becomes 4 ÷ 2, or 2. The result is also halved. Fill in the chart.

Start Factors and Product	Word Statement Card	New Factors and Product	What Happened?
8 ÷ 2 = 4	halve the numerator	4 ÷ 2 = 2	result halved

- Explain that some combinations of division statement card and number card will not work for this activity. An “inappropriate” word card would be a division statement that does not result in a whole number or is not considered a basic division fact.

– Give an example, such as the following.

Division card: 12 ÷ 4 Word card: *halve the numerator*

In this case, halving the numerator yields the statement 6 ÷ 4, which is not a basic division fact (since it results in 1.5).

This word card is rejected and another word card is selected.

Explore

Learners set the game up by placing the 10 number cards face down in an array, and stacking the word statement cards face down.

- A learner selects one card from the number card array and one card from the word card stack.
- First, the learner states the division fact and its result.
- Second, the learner must decide if the word statement card is appropriate for the selected number card. If not, a new word card is selected. (The word cards are replaced in the stack after each use.)
- Third, the learner uses the word statement card to state the new division fact and give the result.
- Fourth, the learner states what happened to the original result after the word statement card is applied.
- Meanwhile, the other learner records all the information on the recording chart.
- Learners then change roles and repeat the process until all of the number cards have been selected.

Closure

Ask learners to explain what happened to the result when they chose *halve the denominator*.

Assessment	***Evaluation***
Learners select a number card. Example: 12 ÷ 4	
Learners select a word card. Ask, “What is the new division fact?” Example: *halve the denominator*	
Ask, “What is the result of the original statement? What is the result after you halve the denominator?”	Success is verified by 12 ÷ 4 = 3 and 12 ÷ 2 = 6.
Ask, “What happened to the result?”	Success is verified by declaring the result is doubled when the denominator is halved.

Adaptations for Various Learners

ELL:

- Demonstrate the meaning of all the word statements.
- Begin by using only two of the word cards: *double the numerator, double the denominator.*
- Later include all five word cards.

Special Needs:

- Demonstrate the meaning of all the word statements.
- Begin by using only two of the word cards: *double the numerator, double the denominator.*
- Later add *halve the denominator.*

Gifted:

- Expand the division statement cards to include 10s and 12s.
- Include other word cards such as *triple the denominator* and *triple the numerator*.

Hit the Target Number

Topic: Arithmetic operations practice
Instructional Mode: Small groups (3 or 4 learners)
Level: 6th grade and up
Time: 20–30 minutes

Pre-assessment

Check understanding of the basic arithmetic operations of addition, subtraction, multiplication, and division.

Materials

- Target bag containing 10 slips of paper with two-digit values written on them (for example, 26, 31, 55, 58, 63, 64, 76, 79, 85, 97)
- Five number cubes
  - Three number cubes with 1 through 6 on their faces
  - Two number cubes with 7, 8, and 9 twice on their faces

Launch

The challenge in this game is to create operations using five given numbers (from a roll of the cubes) to get as close as possible to the target number drawn from the bag.

Demonstrate how the game is played.

- Select a number from the target bag. Let's say it is 31.
- Roll all five number cubes. Let's say the cubes show these five numbers: 2, 4, 6, 8, and 9.
- Explain to learners that they need to construct mathematics statements, using each number only once, to get a result close to or exactly 31.
- In this example, you could add all 5 values for a sum of 29. (2 + 4 + 6 + 8 + 9 = 29)
  - But you could also do a series of operations to get even closer to 31. You could start with 9 – 8 = 1. You've used 9 and 8, and now you have a new number to work with, 1, as well as the remaining three numbers, 2, 4, 6.
  - Next, you create this statement: 2 – 1 = 1. Now you have 1 and 4 and 6 to work with. You see that 4 + 1 = 5 and 6 x 5 = 30. You have just gotten closer to the target number of 31.

Explore

- A small group of 3 or 4 learners can do this activity.
- Explain the directions of Hit the Target Number again to the small group or hand out an activity card with the directions written on it.
- Continue the activity until all 10 values have been drawn from the target bag.
- Have the learners orally explain their expressions. The expressions are usually not easily written statements.

Closure

Ask the learners to think of another number value for the target number, roll the five cubes, and find a close expression. See if they can find more than one expression.

Assessment	***Evaluation***
Give learners a value for the target number.	Success is appropriately combining the arithmetic operations to form an expression close to a target number.
Roll the five number cubes and have the small group work together to find a close or exact expression.	

Adaptations for Various Learners

ELL:

- Demonstrate the actions of selecting a target number and rolling the five number cubes.
- Encourage oral expression and maybe written expressions of part of the process of finding the closest expression.

Special Needs:

- Change the values in the target bag to include lower values, for example in the 10s, 20s, and 30s.
- Roll only two number cubes with 1 through 6 and one number cube with 7, 8, 9 (twice) on the faces.

Gifted:

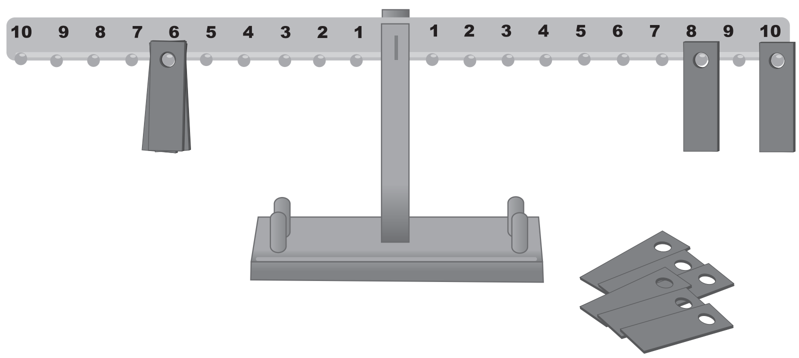
- Change the values in the target bag to include three-digit numbers.
- Have the learners write the expressions that are closest to the target number.
- Have the learners continue exploring until the five number values from the number cubes can yield in an exact match with the target number.

Multiplication Balance

Topic: Multiplication number facts to 81
Instructional Mode: Small group, individual
Level: 3rd grade and up
Time: 20–30 minutes

Pre-assessment
Check understanding of the multiplication concept. Check understanding of place (ones, tens, hundreds). For example, 18 is 1 ten and 8 ones.

Materials
- Math balance device and weights



- Multiplication statement cards to 81 (e.g., 3 x 8 = [ ], 9 x 7 = [ ], etc.)

Launch
Demonstrate how the math balance works by putting weights on the pegs. The number of weights and distance from the fulcrum determine the correct solution.
- Demonstrate how the product of two factors can be found using the math balance device. For example, 3 x 6 = [ ].
  - On the left side, hang 3 weights on the 6 peg. Ask learners to suggest which pegs you should hang weights on, on the right side, to balance the device. Show that hanging 1 weight on the 10 peg and 1 weight on the 8 peg will balance the device. Therefore, 3 x 6 = 18.
- Repeat with 6 x 3 = [ ] to show the commutative property of the multiplication operation.
- Repeat with larger factors to show the need to place more than one weight on the 10 peg. For example, 4 x 7 = [ ].
  In this case, 2 weights are needed on the 10 peg and 1 weight on the 8 peg to balance the device.

Explore

One math balance is sufficient for a learning center. Two children can use the math balance.

- With the statement cards as a guide have learners, in pairs or individually, answer the multiplication number facts using the math balance.
- At first, give learners only the statement cards that have products less than 30.
- Later, have learners use all the statement cards.
- Check if the learners are applying the place value concept correctly (e.g., 27 should be 2 weights on the 10 peg and 1 weight on the 7 peg).

Closure

- Have a learner explain the sequence of steps used to balance the device.
- Ask the learner about the place-value weight.
- Ask the learner about the turnaround multiplication number facts (commutative property of multiplication).

Assessment	***Evaluation***
Select a specific multiplication number fact statement card. Place the weights for this fact on the left side. Example: 6 x 7 = [ ]	
Have a learner show where the weights must be placed on the right side to balance the device.	Success is placing the weights correctly to balance the device. In this example, 4 weights on the 10 peg, and 1 on the 2 peg.
Have the learner also show the turnaround number fact.	Success is showing the commutative property of multiplication by repositioning the weights.

Adaptations for Various Learners

ELL:

- Use diagrams to show how to use the math balance to complete multiplication situations.

Special Needs:

- Use diagrams to show how to use the math balance to complete multiplication situations.
- To build learners' confidence, begin with the multiplication number facts less than 30.
- Later, gradually add multiplication number facts greater than 30.

Gifted:

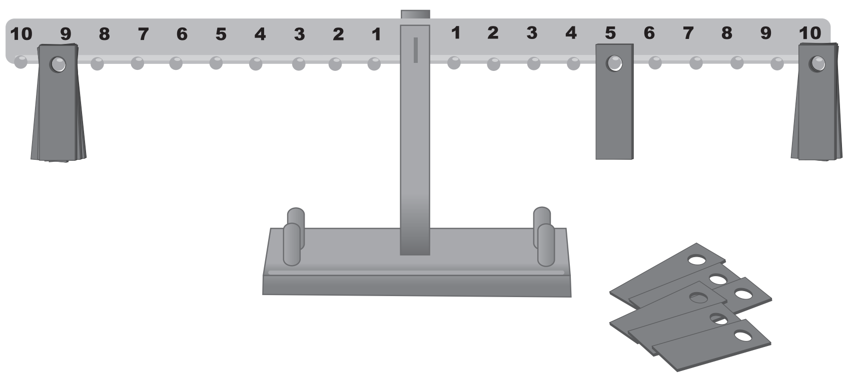
- Include statement cards with 10s, 11s, and 12s as factors in the multiplication number fact.
- Have learners explore other possible configurations of weights to balance the device.

Division Balance

Topic: Division number facts from 81
Instructional Mode: Small group, individual
Level: 4th grade and up
Time: 20–30 minutes

Pre-assessment
Check understanding of the division concept and its relation to the missing multiplication factor.

Materials
- Math balance device and weights



- Statement cards with division number facts from 81 (e.g., 24 ÷ 8 = [ ], 63 ÷ 7 = [ ], 48 ÷ 6 = [ ], etc.)

Launch
Demonstrate how the math balance works by putting weights on the pegs. The number of weights and distance from the fulcrum determine the correct solution.
- Demonstrate how the division number facts can be found on the math balance. For example, 45 ÷ 9 = [ ].
  - On the right side, hang 4 weights on the 10 peg and 1 weight on the 5 peg. On the left side, place weights on the 9 peg until the device balances. Count the number of weights on the 9 peg. It should be 5 weights. Therefore, 45 ÷ 9 = 5.

Explore
One math balance is sufficient for a learning center. Two children can use the math balance in this activity.
- Have learners, in pairs or individually, choose a statement card and then use the math balance to complete the division number fact.
- Check if learners are setting up the situation correctly on the left and right side of the math balance.
- Check for understanding of the tens place-value (10) peg.

- Continue the activity until all the statement cards have been completed.
- Have the small groups or individuals select the five division statement cards they found the hardest.
- Learners should use the math balance for additional practice with these five difficult division number facts.

Closure

- Have a learner explain the process of completing a division number fact on the math balance.
- Check for understanding of place value.
- Check for understanding of the commutative property and division. The division operation does not possess the commutative property.

Assessment	***Evaluation***
Select a statement card with a division number fact. Example: 15 ÷ 5 = [ ] Have learners show these division number facts on the math balance.	Success is correctly placing the weights to balance the device for a beginning division number fact.
Next choose a statement card with a larger division number fact. Example: 63 ÷ 7 = [ ] Have learners show these division number facts on the math balance.	Success is correctly placing the weights to balance the device for a larger division number fact.
Have the learner also state both division number facts.	Success is correctly stating both division number facts.

Adaptations for Various Learners

ELL:

- Use diagrams to show how to use the math balance to complete a division number fact.

Special Needs:

- Use diagrams to show the action of the math balance.
- Begin with division number facts involving 1s, 2s, 3s, and 4s.
- Later gradually add the other division number facts involving 5s, 6s, 7s, 8s, and 9s.

Gifted:

- Include statement cards with 10s, 11s, and 12s values.
- Have learners explore other ways to balance the device for a correct solution to a division number fact.

Measures Of Success

3

Measures of Success

Models of Learning and Instruction

A carefully designed program to help students acquire the basic number facts must include supportive research on instructional intervention and automaticity strategies. In the Overview of this guide, we discuss the basic number fluency research that underlies the interventions detailed in Sections 1 and 2. The protocols included with each section follow the best practices from the available research on acquiring basic number facts. This final section focuses on the measures of success and the path to automaticity with basic number facts.

This guide follows three models of learning and instruction. The first model (used in the protocols) is the sequence of hands-on activities followed by pictorial diagrams, then abstract representation. This model is flexible and allows teachers to adapt the protocols for different grade levels.

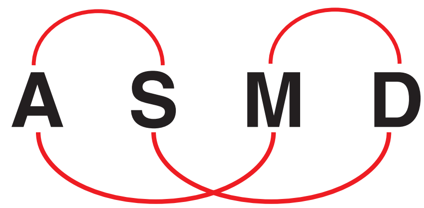
Touch it.	See it.	Think it.
concrete	**pictorial**	**abstract**

The second model presents basic number facts by introducing the concepts of addition, subtraction, multiplication, and division, having learners practice to become confident with the basic number facts, and then practice until they've reached automaticity. This model encompasses a more realistic expectation for acquiring automaticity with the basic number facts. The practice-to-gain-confidence stage allows for identification of conceptual mistakes and the opportunity to use various interventions. It is not realistic to assume that a learner can move directly from understanding the concept of an arithmetic operation to rapid recall of number facts. The learner needs to move from deriving the basic number fact (gaining confidence with his or her abilities to perform the arithmetic operation) to automaticity of recall. The protocols are an important part of the process.

Develop concept ⟶ **Practice with confidence** ⟶ **Recall facts with automaticity**

Another is the program MathFacts in a Flash, which provides individualized practice and feedback for automatic recall. The key to automaticity is practice, and lots of it (Willingham, 2009).

The third model for teaching basic number facts centers on the interrelationships among the four arithmetic operations. These operations build on each other; students can use their knowledge of one to understand another. There is a definite structure to this interrelationship (shown in the diagram), which can enhance learners' grasp of number sense.



The protocols use these interrelationships to explore basic number facts. For example, subtraction and division situations are presented in terms of missing addends and missing factors—addition and multiplication concepts.

Achieving Automaticity Recall

When deciding on a measure of success, teachers should understand that rigid, general goals are not sufficient. Declaring that all multiplication basic number facts must be rapidly recalled by the end of third grade is an unrealistic statement given our knowledge of learning and individual development. Even so, some general principles can be taken into consideration when determining success measures. For instance, in addition, the basic number facts with sums less than 10 are considered easier than those with sums greater than 10. Classroom teachers note that the subtraction basic number facts are more difficult for students to recall than the addition facts. In multiplication, according to classroom teachers, learners find the 5 number facts fairly easy to recall, while those with factors 7, 8, or 9 are harder to recall rapidly and accurately. The division number facts are generally more difficult to recall than the multiplication number facts. Most learners use the related multiplication number fact to recall the division number fact. Therefore, the rapid recall of multiplication number facts is most crucial for success in recalling basic division number facts.

Even with students' different learning rates and difficulties in acquiring basic number facts, we can construct reasonable expectations for practice recall at a rate of 3 or 4 seconds and automaticity recall at a rate of 2 or 3 seconds. Naturally, these broad expectations will not align with all learners. Teachers should adapt them for ELL, special needs, or gifted students.

Level	Concept Development	Practice Recall Rate: 3 or 4 Seconds	Automaticity Recall Rate: 2 or 3 Seconds
early grade 1	addition to 10		
mid grade 1	subtraction from 10	addition to 10	
late grade 1	addition to 18 subtraction from 18	subtraction from 10	
early grade 2		addition to 18	
late grade 2		subtraction from 18	addition to 18
late grade 3			subtraction from 18
late grade 3	multiplication to 81		
early grade 4	division from 81		
mid grade 4		multiplication to 81	
late grade 4		division from 81	multiplication to 81
late grade 5			division from 81

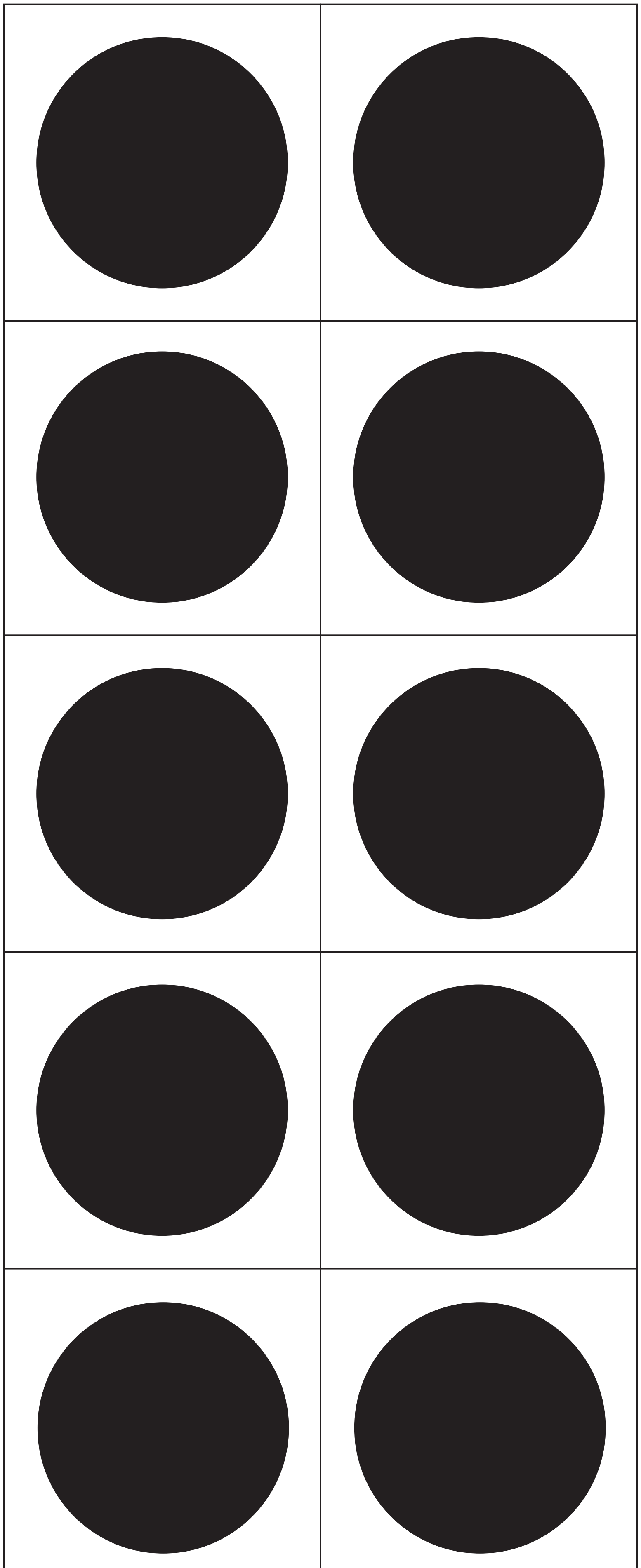
Success is built by moving from small steps to larger steps. A list of *Tips for Success* appears in both the *Early Numerical Skills (Grades 1–3)* and *Extending Numerical Skills (Grades 3–8)* sections. These tips may be considered the small steps leading to success. From these tips the three models of instruction can be gleaned.

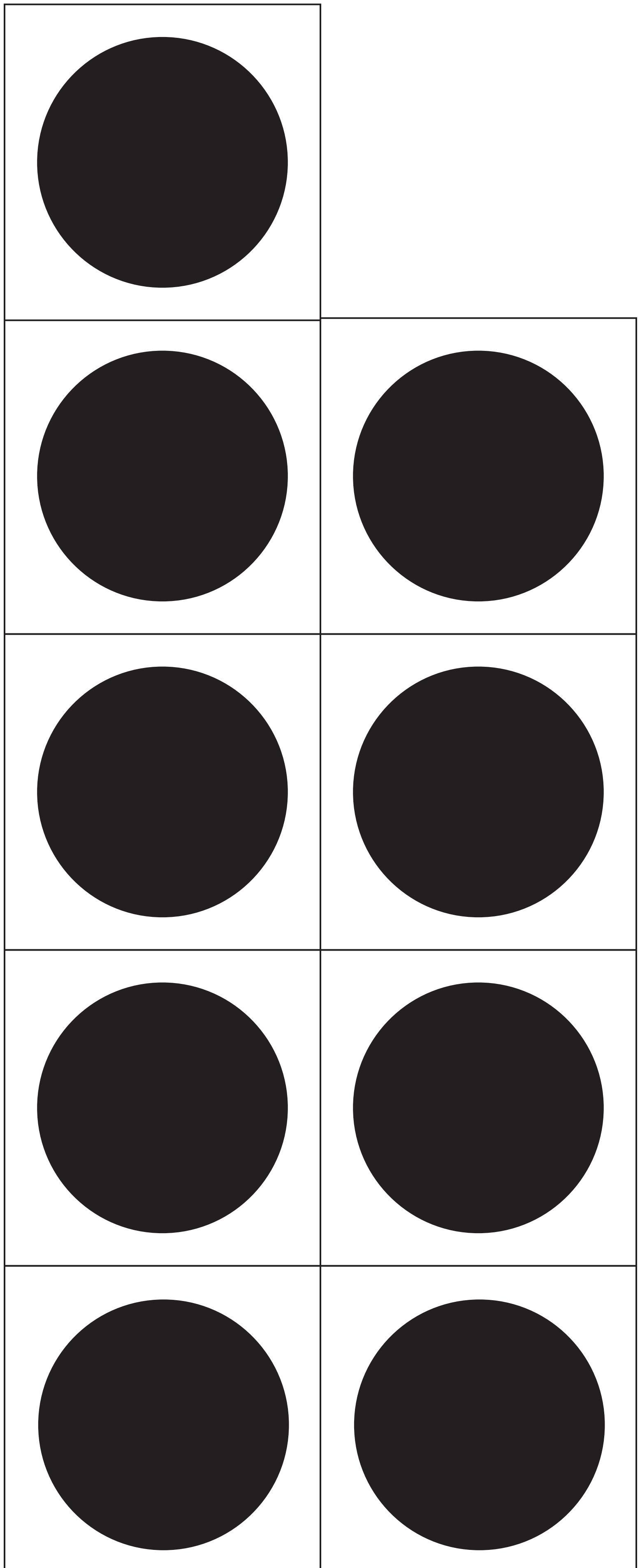
- Keep the mode sequence as the keystone principle of instruction: touch it, see it, think it (hands-on to pictorial to abstract).
- Develop the concept, move to practicing with confidence, and then to practicing to reach automaticity.
- Build the learning process from the interrelationships of addition, subtraction, multiplication, and division.

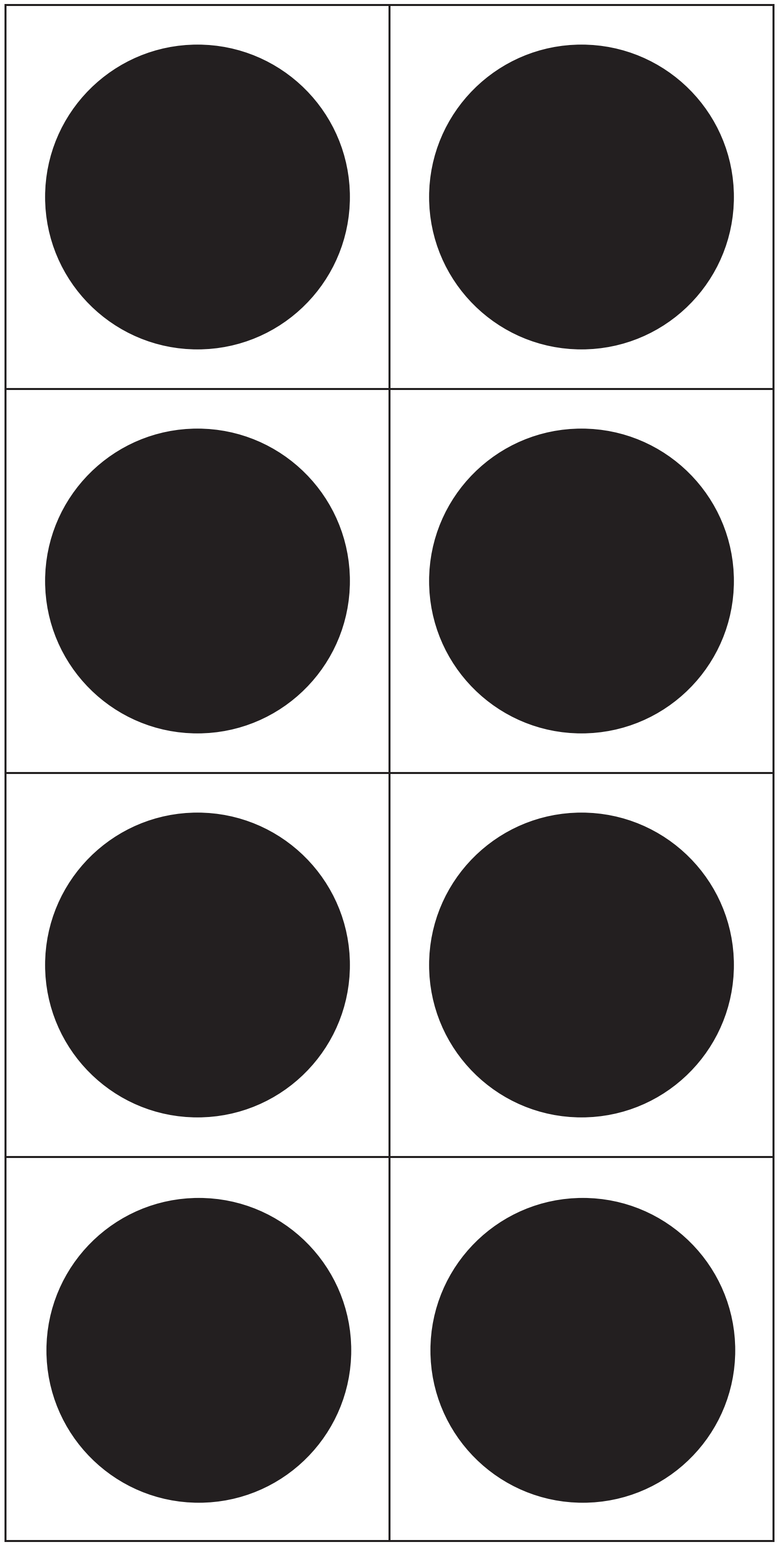
A successful numeracy development program focuses on achieving automaticity recall of basic number facts. However, an educator should always remember that resting on past successes or believing past actions will be successful in the future can lead to unsatisfactory results. Constantly looking for ways to enhance the interventions for developing numeracy is a necessity. Remember to continuously search out other paths to success, rather than rest. As Will Rogers liked to say, *"Even if you're on the right track, you'll get run over if you just sit there!"*

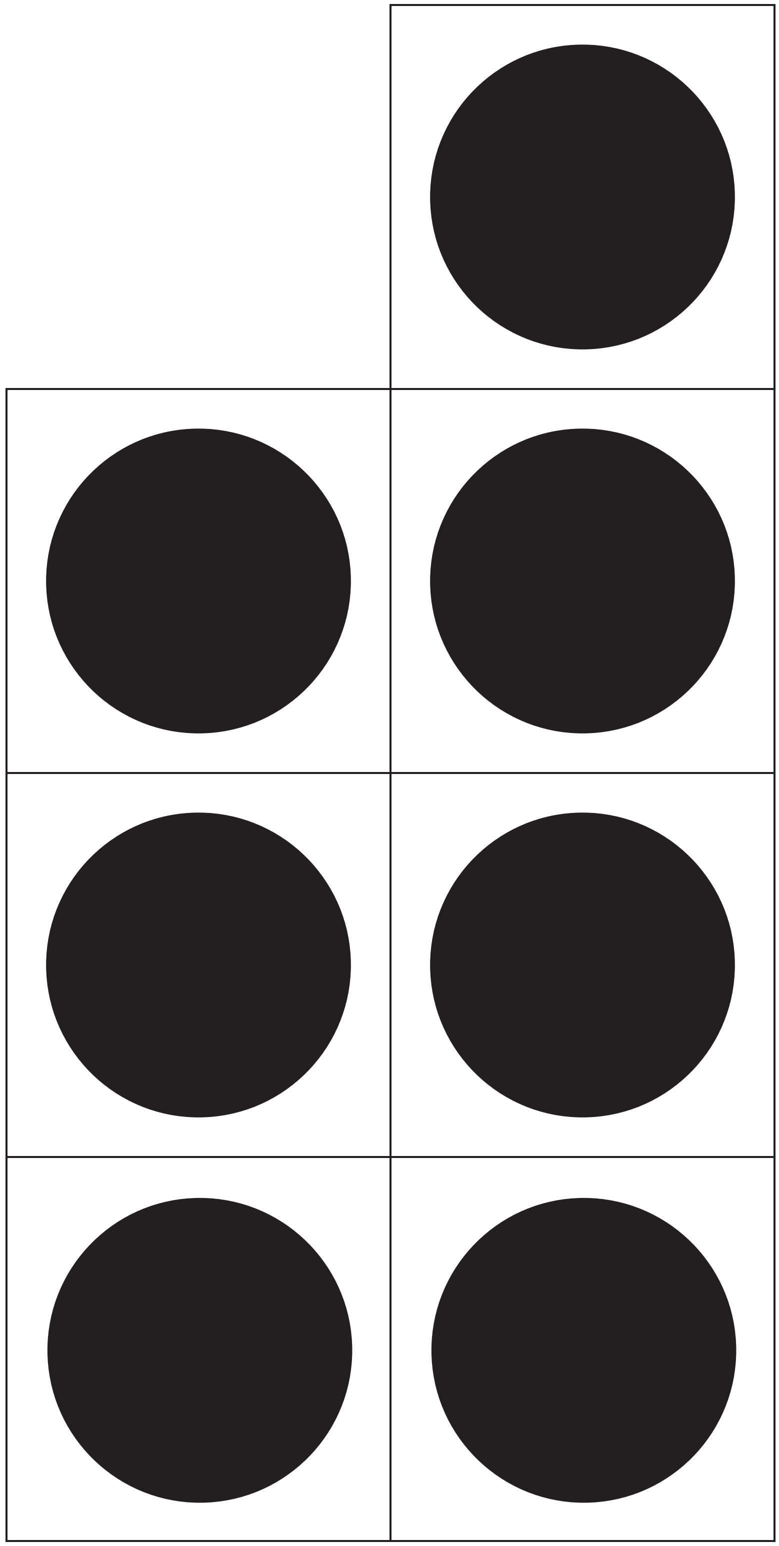
We hope that the strategies and protocols we've presented here inspire you to develop ever more effective *(and fun!)* activities for teaching the basic arithmetic facts to your students.

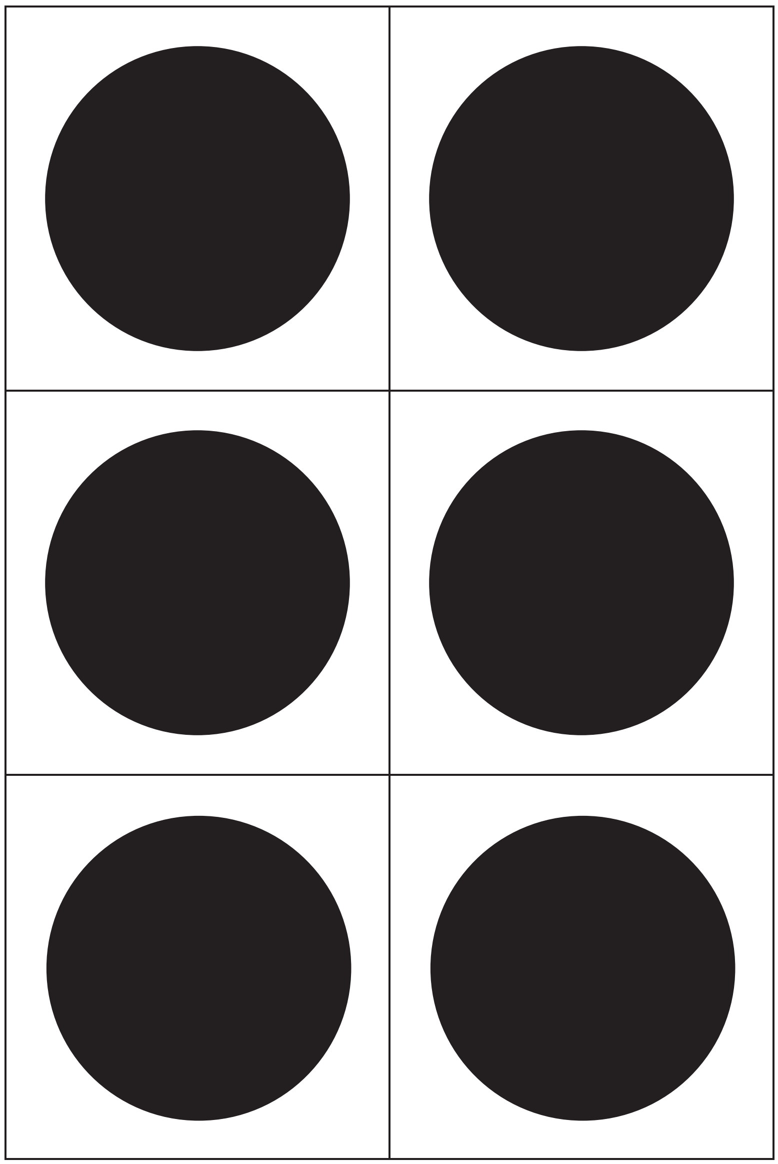
Appendix

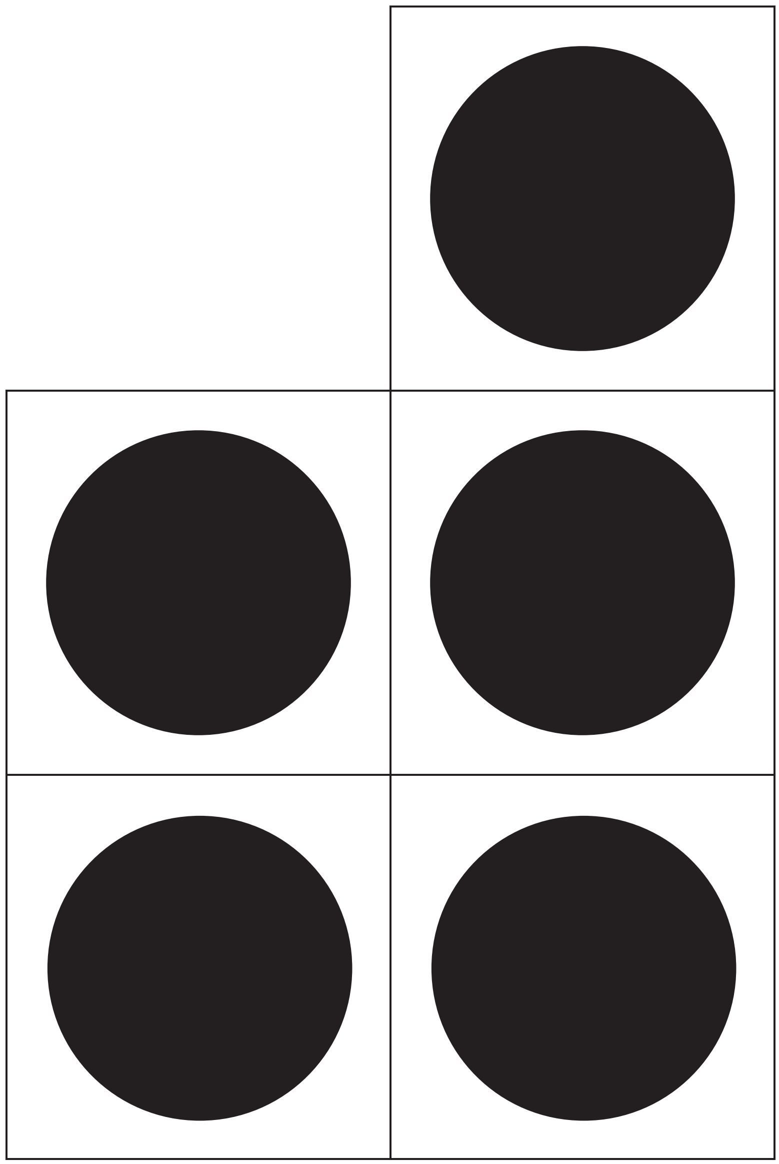


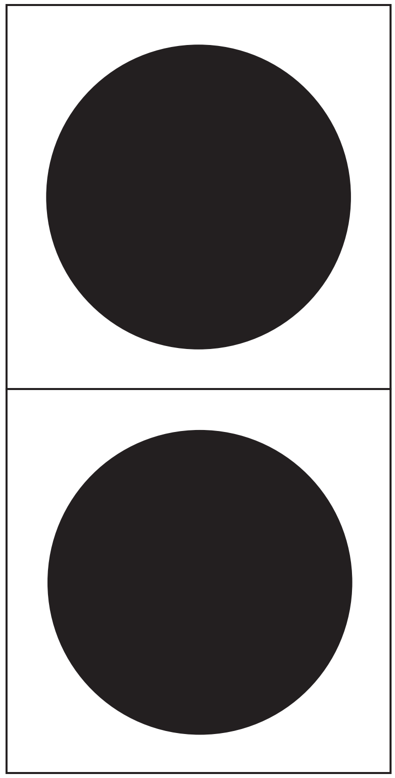


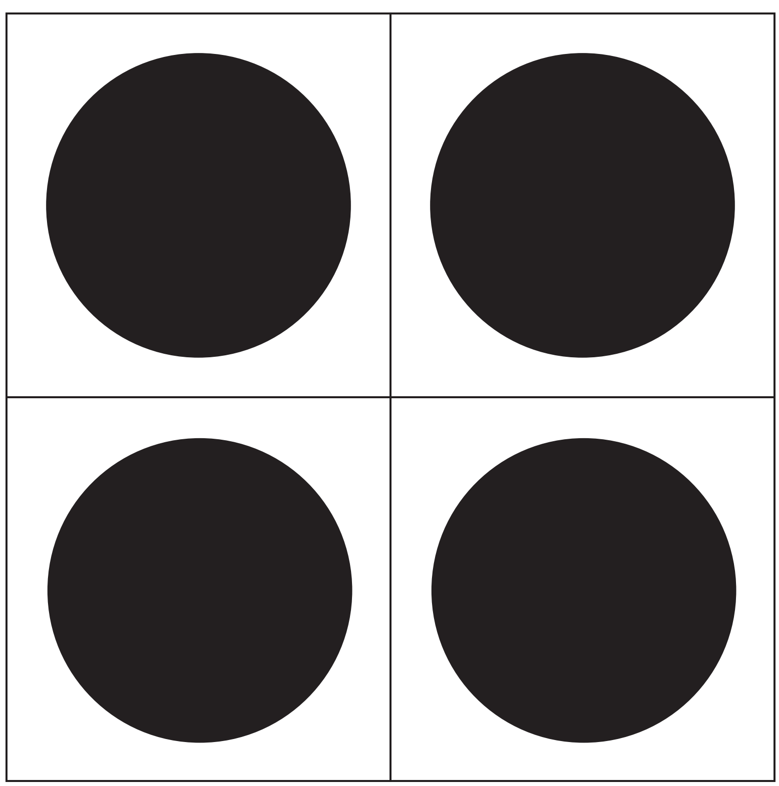


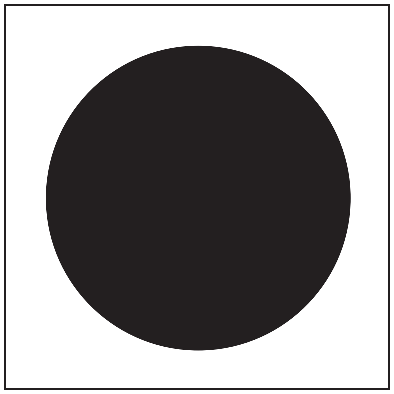


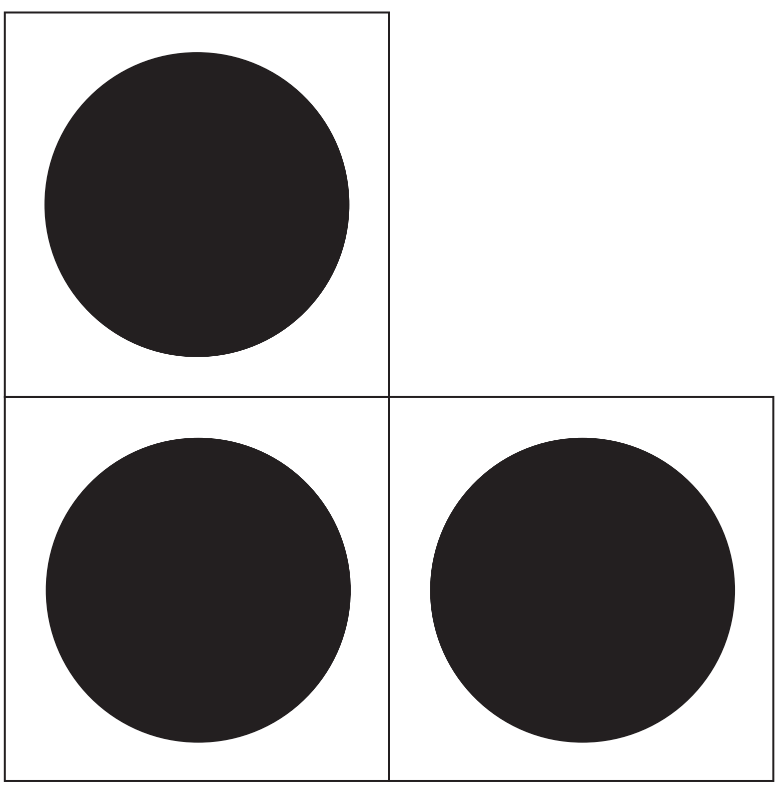












	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0										
1										
2										18
3										27
4										36
5										45
6										54
7										63
8										72
9			18	27	36	45	54	63	72	81
10										
11										
12										

Start Factors and Product	Word Statement Card	New Factors and Product	What Happened?

Starting Division Fact and Result	Word Statement Card	New Division Fact and Result	What Happened?

References

Axtell, P. K., McCallum, R. S., Bell, S. M., & Poncy, B. (2009). Developing math automaticity using a classwide fluency building procedure for middle school students: A preliminary study. *Psychology in the Schools, 46,* 526–538.

Baroody, A. J. (1985). Mastery of basic number combinations: Internalization of relationships or facts? *Journal of Research in Mathematics Education, 16*(2), 83–98.

Battista, A. (1999). The mathematical miseducation of America's youth. *Phi Delta Kappa*.

Bryant, D. P., Bryant, B. R., Gersten, R., Scammacca, N., & Chavez, M. M. (2008). Mathematics intervention for first- and second-grade students with mathematics difficulties: The effects of tier 2 intervention delivered as booster lessons. *Remedial and Special Education, 29*(1), 20–32.

Garnett, K., & Fleischner, J. E. (1983). Automatization and basic fact performance of normal and learning disabled children. Learning Disability Quarterly, 6(2), 223–230.

Gersten, R., Beckmann, S., Clarke, B., Foegen, A., Marsh, L., Star, J. R., et al. (2009). *Assisting students struggling with mathematics: Response to intervention (RTI) for elementary and middle schools* (NCEE 2009-4060). Washington, DC: U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. Retrieved July 16, 2009, from http://ies.ed.gov/ncee/wwc/publications/practiceguides/

Gersten, R., & Chard, D. (1999). Number sense: Rethinking arithmetic instruction for students with mathematical disabilities. *The Journal of Special Education, 33*(1), 18–28.

Gersten, R., Jordan, N., & Flojo, J. (2005). Early identification and interventions for students with mathematics difficulties. *Journal of Learning Disabilities, 38*(4), 293–304.

Hartnedy, S. L., Mozzoni, M. P., & Fahoum, Y. (2005). The effect of fluency training on math and reading skills in neuropsychiatric diagnosis children: A multiple baseline design. *Behavioral Interventions, 20*, 27–36.

Hill, N. M., & Schneider, W. (2006). Brain changes in the development of expertise: Neurological evidence on skill-basedadaptations. In K. A. Ericsson, N. Charness, P. Feltovich, & R. Hoffman (Eds.), *Cambridge handbook of expertise and expert performance.* New York: Cambridge University.

Howell, K. W., & Lorson-Howell, K. A. (1990). What's the hurry? Fluency in the classroom. *Teaching Exceptional Children, 22*, 20–23.

Isaacs, A. C., & Carroll, W. M. (1999). Strategies for basic-facts instruction. *Teaching Children Mathematics, 5*(9), 508–515.

Logan, G. D. (1988). Toward an instance theory of automatization. *Psychological Review, 95*(4), 492–527.

National Mathematics Advisory Panel. (2008). *Foundations for success: The final report of the national mathematics advisory panel.* Washington, DC: U.S. Department of Education.

Pegg, J., Graham, L., & Bellert, A. (2005). The effect of improved automaticity of basic number skills on persistently low-achieving pupils. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th conference of the international group for the psychology of mathematics education,* Vol. 4 (pp. 49–56). Melbourne: PME.

Pellegrino, J. W., & Goldman, S. R. (1987). Information processing and elementary mathematics. *Journal of Learning Disabilities, 20*, 23–32, 57.

Rohrer, D., Taylor, K., Pashler, H., Wixted, J. T., & Cepeda, N. J. (2005). The effect of overlearning on long-term retention. *Applied Cognitive Psychology, 19*, 361–374.

Willingham, D. T. (2004, Spring). Practice makes perfect—But only if you practice beyond the point of perfection. *American Educator.* Retrieved July 1, 2009, from www.aft.org/pubs-reports/american_educator/spring2004/cogsci.html.

Willingham, D. T. (2009). *Why don't students like school? A cognitive scientist answers questions about how the mind works and what it means for the classroom.* San Francisco: Jossey-Bass.

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