Orange You Glad I *Did* Say "Fraction Division"?

Students explore multiplicative comparisons and the meaning of remainders using their own concrete representations, including orange wedges.

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When teaching division of fractions, teachers need to address issues and tensions created by trying to accomplish two important goals. The first goal is to let students make sense of division of fractions on their own by working individually and in small groups, using concrete or pictorial representations, inventing their own processes, and presenting and justifying their answers and processes to one another. The second goal is to help students develop their understandings of division of fractions on a deeper level, as a multiplicative comparison, so that they develop proportional thinking.

When comparing two quantities or numbers multiplicatively, a ratio is involved. To help students develop their ability to compare fractions multiplicatively, we need to encourage them to think beyond iterative strategies such as repeated addition or repeated subtraction. The teacher may have to intervene at crucial points to redirect the thinking of students to focus on multiplicative relationships among fractions.

MAKING SENSE OF DIVISION OF FRACTIONS ON THEIR OWN

In this section, we illustrate ways in which four middle school students, solving division of fraction problems in the context of servings, made sense of this operation on their own. The examples come from a larger study by the second author (Day 2010). Because the purpose of her study was to understand how students justified their strategy and solution to their peers, the teacher/researcher listened carefully but did not intervene. The teacher and students agreed that showing one's mathematical thinking included more than producing a simple list of steps or writing an algorithm. Students communicated their thoughts through graphic representations, symbols, and words. To justify their reasoning, students applied three guiding principles:

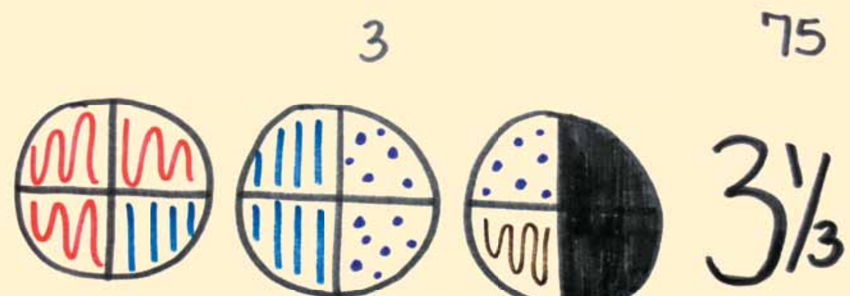
1. Convince themselves by devising personally meaningful solutions to problems

2. Convince others by communicating their understanding through graphic representations, words, and symbols
3. Make sense of other students' justifications to raise challenges if disagreements occurred

Story problems that are set in the context of finding out how many servings can be formed with a given quantity when the size of the serving is known lend themselves to the use of measurement interpretation of division and the use of repeated subtraction to find the answer. This setting has the advantage of allowing students to use concrete or pictorial representations and find their own strategies (Kribs-Zaleta 2008).

The simplest case of a measurement interpretation of division occurs when the size of the serving fits exactly into the amount being distributed. For example, if a serving is 3/4 of an orange and there are 4 1/2 oranges, exactly 6 servings are found. A more difficult situation occurs when the size of the serving does not fit a whole number of times into the amount and a fractional remainder is leftover. Students working on problems with a fractional remainder usually struggle to make sense of the remainder in terms of the unit used to measure (serving), rather than in terms of the objects used (oranges or cupcakes). We focus first on the interactions of four middle school girls—Taylor, April, Rosalee, and Rebecca (pseudonyms)—as they worked on the following problem.

Fig. 1 Taylor's graphic representation for her solution was helpful, but her explanation remained incomplete because the remainder was not labeled.



By showing 1/3 of a serving, Taylor moved the conversation forward by addressing what constituted a "whole" and how that related to the remainder.

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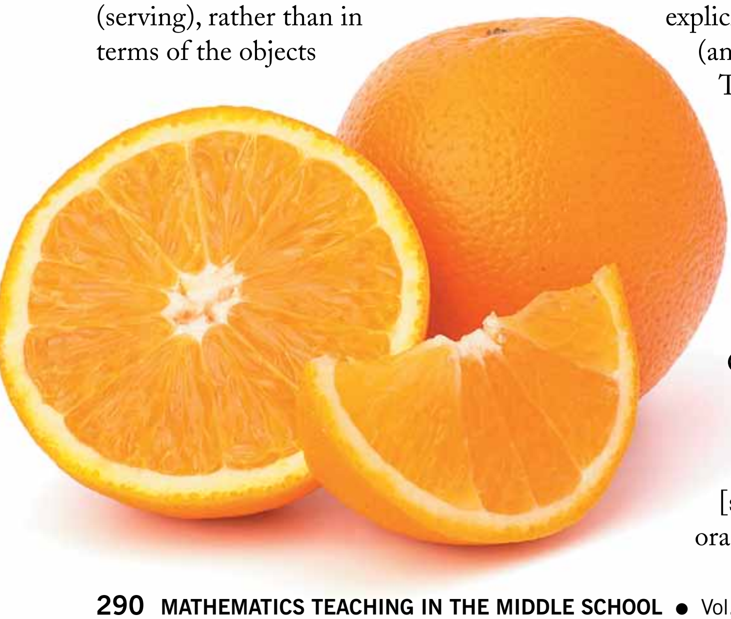


THINKSTOCK

> A serving is 3/4 of an orange. There are 2 1/2 oranges. How many servings (including parts of a serving) can be made?

When the students communicated their thinking to one another, confusion occurred because they did not explicitly define their "whole" (an orange or a serving). This confusion caused disagreement as they worked toward the final solution to the problem.

Taylor: Suppose you have 2 1/2 oranges. One . . . 2 . . . 3. I just did 3 because I thought it would be easier for me. So this is 1 serving [she shaded in 3/4 of an orange] and this is another [she shaded in another 3/4 of an orange] . . . this is another serving [she shaded in 3/4 of an orange] . . . and yea. So this is technically not there. [Taylor pointed to the last 1/2 of the third orange.] You're not supposed to . . . so there's 3 servings right there. There's 1 leftover. And there's out of 3 possible. . . . So you know how you need 3 like 3/4. . . .

Rosalee: Yea.

Taylor: . . . to get 1 serving. So that's like seventy-five cents, but instead of like doing that you have 1 out of 3 possible. So I put 3 1/3. 'Cause it's 1/3 leftover.

The unit specified in the question is a serving. Students correctly figured out the number of complete servings by using repeated subtraction of servings, but they struggled with interpreting the remainder. The remainder is 1/4 of an orange but is 1/3 of a serving.

Taylor used her graphic representation to show the remaining amount, but she was not always clear in her use of symbols and words when relating the remaining part to a whole unit, such as servings (see **fig. 1**). Therefore, at times she struggled to communicate effectively with others when trying to explain her strategy.

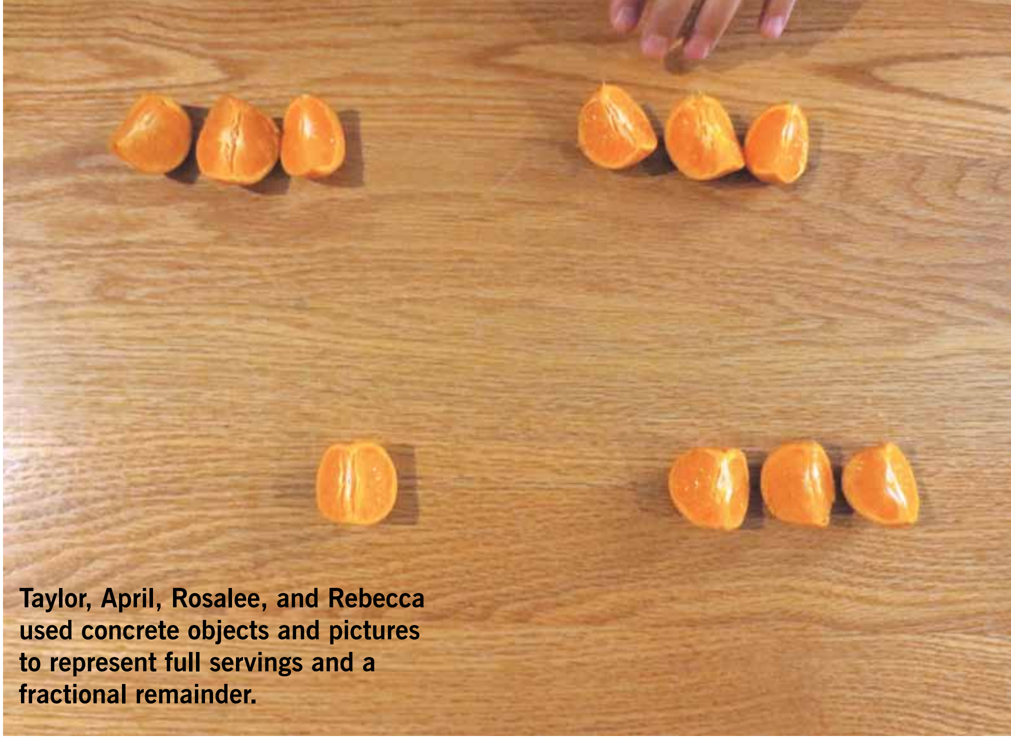
Taylor's solution of 3 1/3 was correct, but she did not specify that it was 3 1/3 *servings*. She said that there was 1 leftover. From a preliminary written survey and her oral interview with the teacher, it was evident that Taylor understood that the remaining amount represented 1/4 of an orange. However, she did not specify that the remaining amount represented 1/4 of an orange when she explained it to her group.

Later in the discussion, Taylor continued to try to convince her group by being explicit that the remainder was not 1/4 by saying, "But it's not a fourth of a serving." The girls had mentioned servings before, but this was the first time they discussed a fraction of a serving. It may have helped her group understand if Taylor would have explained that a serving was 3/4 of an orange and that it took three of these 1/4 oranges to make a full serving.

The students were using graphic representations to show the remaining amount as well as symbols and words. However, they needed to be more explicit when using their symbols and words to explain what the remaining amount represented in units as a whole, so they could communicate when they were referring to different wholes.

Later in an interview with the teacher, Taylor stated that she needed to explain her answer more clearly.

> Like the problem said, you were supposed to look at the serving and not the oranges. Well, I never said that in my answer. And I think if I would have explained that better they would have come to know how I got the answer.



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Taylor, April, Rosalee, and Rebecca used concrete objects and pictures to represent full servings and a fractional remainder.

Although Taylor and her group realized that they needed to clearly state the unit, this error of not making the unit explicit was persistent throughout the study. In the following sections of this article, we suggest ways in which teachers can help students step back so that they can develop their mathematical understandings of division of fractions at a deeper level.

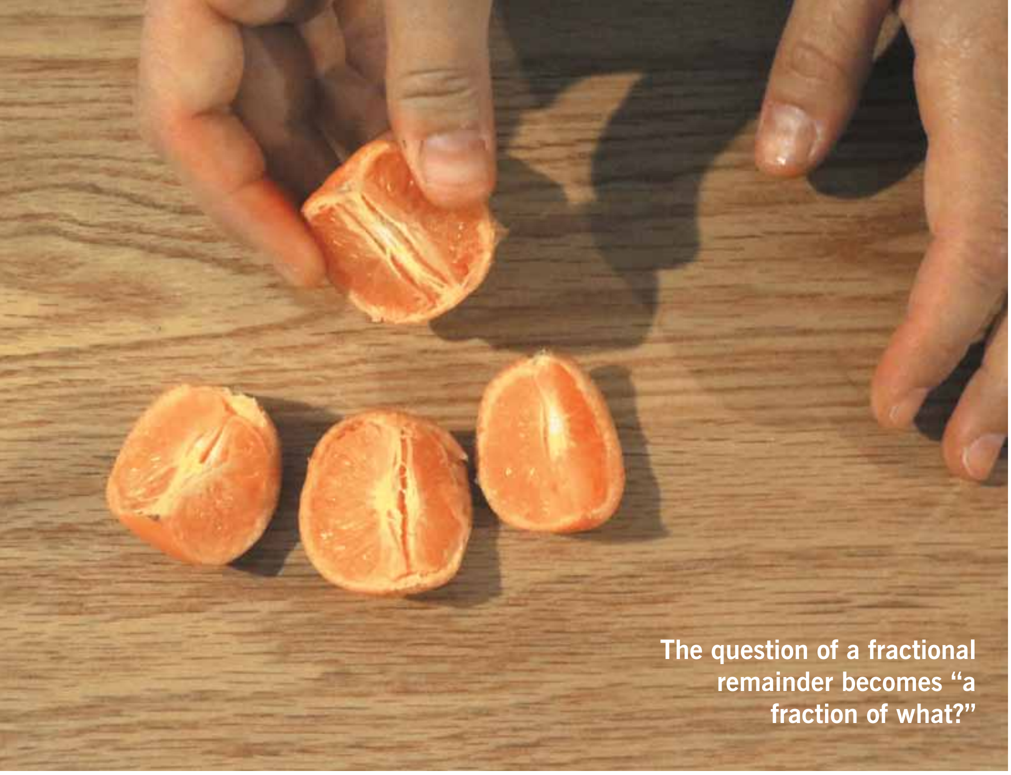
CONNECTING DIVISION OF WHOLE NUMBERS AND FRACTIONS

By stepping back to consider division problems with whole numbers and interpreting the result as a fraction, students establish an important connection that is not always present. Some students will notice on their own that the same numbers appear in a division problem, such as 7 ÷ 4 appearing in the answer as 7/4. On their own, or with a teacher's guidance, students need to realize that the fact that the same numbers appear in the original division problem and in the answer expressed as a fraction 7 ÷ 4 = 7/4 is not a coincidence. Each person's portion in a distribution situation can be predicted by the (multiplicative) relation between objects and participants. This realization is one example of the close relationship that exists between proportions and fractions and how the concepts are intertwined (Streefland 1991, p. 130). This realization will be a stepping-stone later when students think in terms of ratio when dividing fractions.

One way that teachers may help students understand the meaning of a remainder in a division of fractions problem is to direct their attention to similar problems using whole numbers. For example, consider the following problem.

> A serving is 3 cupcakes. There are 7 cupcakes. How many servings are there?

One answer is 2 servings and 1 cupcake as remainder. Students need to interpret the remainder in terms of the serving. They must realize that they need to divide the remainder by the size of the serving; in other words, the remainder also needs to be divided by the divisor. If a serving is 3 cupcakes, then 1 cupcake is 1/3 of a serving. Therefore, another answer is 2 1/3 servings. When students write the answer to 7 ÷ 3 as 2 R1, they need to realize that the 2 refers to how many servings, not cupcakes, but that the remainder 1 refers to cupcakes.


The question of a fractional remainder becomes "a fraction of what?"

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A TEACHABLE MOMENT: THE REMAINDER AS A QUOTIENT

While Taylor and April were working on the following problem, they made explicit for the first time in the study a way to compare the remainder with the serving multiplicatively.

> Adam has been serving 2/3 cup of lemonade to each student. If he has 1 1/2 cups of lemonade left, how many students can still get lemonade? How much of a serving will the last student get?

To solve the problem, both girls added

$$\frac{2}{3}+\frac{2}{3}=1\frac{1}{3}.$$

Taylor performed the calculation in her head, and April used a picture. Then they found the difference between this value and 1 1/2. They both thought the answer was 2 whole servings and 1/6 of a serving. However, Taylor at one point said 1/6 *of a cup* and then a minute or so later she said 1/6 *of a serving*. She changed what she said without even realizing it. When the teacher reminded them about the orange problem, they concluded that they were mixing up the units. The girls realized that the 2 represented student servings whereas the 1/6 represented cups of lemonade. Taylor stated:

> That would be cups of lemonade leftover. And then we have to figure out if it takes . . . if it takes 2/3 and you have 1/6 leftover. So if you had 1/6, and it needs to go into 2/3 of a cup of lemonade, how would we do that?

Taylor's statement "That would be cups of lemonade leftover" indicates that she realized that the remaining 1/6 represented a fraction of a cup, not 1/6 of a serving. April suggested how to deal with the remainder to express it as part of a serving when she asked, "One-sixth divided by 2/3?" The girls then contemplated the idea of dividing the remainder by the serving, but could not quite remember the procedure for dividing fractions, so they could not follow this approach.

This situation is one example in which lack of computational skill can prevent further conceptual exploration. A teacher can approach this situation in several ways. One strategy is for the teacher to note the situation and ask students to place their thoughts on hold, then help them remember or relearn how to divide fractions, so that they can continue the problem. One way to divide fractions that highlights the connection of ratio with division is using a common denominator. Their problem

$$\frac{1}{6}\div\frac{2}{3}$$

is equivalent to

$$\frac{1}{6}\div\frac{4}{6}.$$

In this division problem, we need to compare, by forming a ratio, the number of pieces in the group that represents 1/6 with the number of pieces in the group that represents 4/6. Because the pieces in each group are the same size, the result will be equal to the ratio of the number of pieces in each group, that is, the ratio of the numerators, 1 ÷ 4, or 1/4.

Another strategy is to take advantage of appropriate technology. A teacher can encourage students to use an inexpensive tool like a fraction calculator to explore their ideas. The context, with some teacher's guidance, will help students interpret the result given by the calculator.

Later, the teacher can come back and ensure that students know and understand how to divide fractions even when they do not have access to a calculator. However, for the time being, forgetting the procedure should not deter students from developing their conceptual understanding. Using the fraction calculator's division key (÷), the girls could have computed

$$1/6 \div 2/3 = 1/4.$$

The important thing at this point is for teachers to help students continue thinking about the comparison of two fractions (the remainder and the serving in this case) in multiplicative terms.

RECONNECTING REPEATED SUBTRACTION TO DIVISION

The initial approach used by Taylor and her teammates was to use the serving as a measurement unit and then to do repeated subtraction or repeated addition to figure out how many servings of lemonade could be formed. Students need to realize that repeated subtraction can also be done in one step by using division.

They can refer first to a familiar situation with whole numbers, such as how many pairs can be formed with six students. Students can easily see that the problem can be solved by repeatedly subtracting (2 from 6) or by dividing (6 ÷ 2). Students can then go back to the situation with fractions and solve

$$2\frac{1}{2} \div \frac{3}{4}$$

with a fraction calculator using the ÷ key. After simplifying, the calculator will display the answer as 3 1/3, the same number students obtained using their own method. Students need to remember that the 1/3 refers to servings of lemonade. Some fraction calculators also have the feature of division with remainder. For example, students using the ÷R key on a Casio® Fraction Mate calculator would obtain the answer 3 R 0.25. Students can recognize that 0.25 = 1/4 and realize that this remainder refers to oranges.

In discussing the power of mathematical notation, Hiebert and Behr (1988) point out that mathematical symbolism can take students beyond their level of conceptual understanding. When students use calculators, however, this case becomes even more obvious because students can easily find answers that they do not understand. Therefore, we need to take the time to ensure that students understand why the different operations and the corresponding keystrokes were chosen and what the answer means. In the example above, students need to relate the two different answers 3 1/3 and 3 R 1/4 and clearly state what each number—3, 1/3, and 1/4—represents in terms of servings and oranges.

To deepen students' understanding of the remainder as part of an orange or as a fraction of a serving, the teacher may point to the relation of division with multiplication. To get the original dividend from the answer by using the divisor as a factor, students need to realize that in one case they would write

$$3\frac{1}{3} \times \frac{3}{4} = 2\frac{1}{2},$$

whereas when using the remainder, they would need to write

$$3 \times \frac{3}{4} + \frac{1}{4} = 2\frac{1}{2}.$$

HELPING STUDENTS MAKE THE TRANSITION

To develop their proportional thinking, students need to shift from the use of composed-unit strategies, such as iterating and partitioning, to multiplicative comparisons (Lobato and Ellis 2010, p. 69). As illustrated here, this transition is not easy for students. Teachers need to be alert to recognize situations in which asking students the right question, providing them with an appropriate tool, or pointing to a similar connection can help students make the transition from additive comparisons to multiplicative comparisons in the context of the division of fractions.

REFERENCES

Day, Melina M. 2010. "Middle School Mathematics Students' Justification Schemes for Dividing Fractions." Unpublished PhD diss. Arizona State University, Tempe, Arizona.

Hiebert, James, and Merlyn Behr, eds. 1988. *Number Concepts and Operations in the Middle Grades*. Vol. 2, Research Agenda for Mathematics Education. Reston, VA: National Council of Teachers of Mathematics.

Kribs-Zaleta, Christopher M. 2008. "Oranges, Posters, Ribbons, and Lemonade: Concrete Computational Strategies for Dividing Fractions." *Mathematics Teaching in the Middle School* 13 (April): 453–57.

Lobato, Joanne, and Amy B. Ellis. 2010. *Developing Essential Understanding of Ratios, Proportions, and Proportional Reasoning: Grades 6–8*. Reston, VA: National Council of Teachers of Mathematics.

Streefland, Leen. 1991. *Fractions in Realistic Mathematics Education: A Paradigm of Developmental Research*. Dordrecht, The Netherlands: Kluwer Academic Publishers. doi:http://dx.doi.org/10.1007/978-94-011-3168-1

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