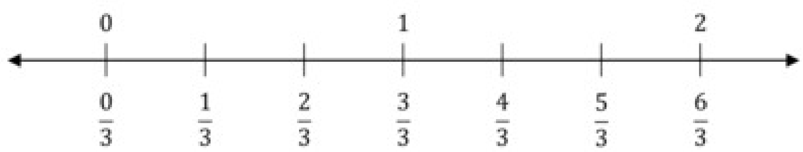
7.NS, 8.NS Repeating or Terminating?

Task

Tiffany said,

I know that 3 thirds equals 1 so $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1$.



I also know that $\frac{1}{3}=0.333\dots$ ***where the 3's go on forever. But if I add them up as decimals, I get*** $0.999\dots$.

$$\begin{array}{r} 0.333\dots \\ 0.333\dots \\ +0.333\dots \\ \hline 0.999\dots \end{array}$$

I just added up the tenths, then the hundredths, then the thousands, and so on. What went wrong?

a. Write $0.999\dots$ in the form of a fraction $\frac{a}{b}$ where a and b are whole numbers. Are Tiffany's calculations consistent with what you find? Explain.

b. Use Tiffany's idea of adding decimals to write $\frac{1}{3}+\frac{1}{6}$ as a repeating decimal. Can this also be written as a terminating decimal?

IM Commentary

The purpose of this task is to understand, in some concrete cases, why terminating decimal numbers can also be written as repeating decimals where the repeating part is all 9's. Terminating decimals also have a more familiar ''repeating'' representation where the repeating part is all 0's: for example 1/2 = 0.5000$\ldots$. Students can benefit from work on this problem in both the seventh and the eighth grade. In the seventh grade, the emphasis should be on checking that the mathematics in Sarah's calculations is correct and so 1 and 1/2 have both a terminating and an eventually repeating decimal expansion. In the eighth grade, students need to be able to convert repeating decimals into fractions and this is a starting point for that work.

A nagging problem remains in many people's minds with the equality

$$1 = 0.999\ldots.$$

Since we cannot readily conceive of infinitely many 9's, it is easy to conclude that the right hand side will never reach the left: indeed this is only possible if it goes on forever. The difficulty of using our intuition for these infinite expansions makes this concept particularly hard for many students. For this reason, teachers should be prepared for student questions about decimals ending with infinitely many 9's and the ideal time for working on this task would be when Tiffany's reasoning and worries arise in student work. Teachers should also be prepared to produce many different ways of explaining the identity, several of which are included in the solutions below.

Solutions

Solution: 2 Two further approaches for part (a)

In order to write 0.999$\ldots$ as a fraction, we begin by studying the decimal form of $\frac{1}{3}$:

$$\frac{1}{3} = 0.333\ldots$$

What this means is that if we add 3/10, 3/100, 3/1000, and so on forever, we will get 1/3. To see why this is the case, note that

$$\frac{1}{3} - \frac{3}{10} = \frac{10}{30} - \frac{9}{30}$$
$$= \frac{1}{30}.$$

So if we just take the first decimal in 0.333… we fall 1/30 short of 1/3. If we take the first two decimals in 0.333… we find

$$\frac{1}{3} - \frac{33}{100} = \frac{100}{300} - \frac{99}{300}$$
$$= \frac{1}{300}$$

so we fall 1/300 short. This pattern continues: 1/3 - 0.333 gives 1/3000 and so on. When we take *all* of the decimal places of 0.333…, however, we obtain 1/3.

This reasoning can help with writing 0.999… as a fraction. The natural candidate is the fraction 3/3 since 0.999… is 3 × 0.333… which is the same as 3 × 1/3. To see that 3/3 and 0.999… are the same, we can argue as in the previous paragraph although the numbers are simpler this time. If we take the difference of 3/3 and 0.9 we get 1/10. When we take the difference of 3/3 and 99/100 we get 1/100. Since the 9's in 0.999… go on forever, this means that the amount left over is less than 1/10, 1/100, 1/1000, and so on forever: since the difference 3/3 - 0.999… is not negative, it must be 0. So 0.999… = 3/3.

All of Tiffany's calculations are correct. There are two different decimal representations for the rational number 3/3. It can be written as 1.000… or as 0.999…. The representation of 3/3 as 0.999… is not usually used but is neither simpler nor more complex than 1.000…. For any decimal number, we need to know *every* decimal place in order to locate the number accurately on a number line. The reason why writing 1 for 3/3 seems simpler than writing 0.999… is that 1 is shorthand for 1.000…. One of the drawbacks of writing fractions as decimals is that simple fractions, like 1/3 or 2/7, have infinite decimal expansions. The behavior of repeating 9's, discovered by Tiffany, is another oddity of decimal expansions for rational numbers.

Alternatively, we can use the decimal expansion of 1/9. We know that 1/9 = 0.111… and if we multiply both sides by 9 this gives 9/9 = 0.999… and we have written 0.999… as a fraction. This is essentially Tiffany's reasoning applied to the decimal expansion of 1/9 instead of 1/3.

Solution: 1 Seventh grade explanation

a. One way to write 0.999… as a fraction exploits the fact that the 9's in the decimal go on forever: if we multiply 0.999… by 10, there will still be infinitely many 9's to the right of the decimal and we can cancel them all out by subtracting. Let x = 0.999…. Then

$$\begin{aligned} 10x &= 9.999\ldots \\ &= 9 + 0.999\ldots \\ &= 9 + x. \end{aligned}$$

Subtracting x from both sides gives $9x$ = 9 and so x = 9/9.

Both methods indicate that 0.999… is equal to 1. This is correct. On the one hand, 0.999… can not be greater than 1 because of the structure of the decimal system: all of the places to the right of the ones place can not add up to more than one. On the other hand, if we add any positive quantity to 0.999… then the sum exceeds 1. This means that 0.999… cannot be smaller than 1. Since it is neither greater than nor smaller than 1, 0.999… is equal to 1.

b. We can check, using the division algorithm, that 1/3 = 0.333… and 1/6 = 0.1666…, where the 3's in 1/3 and the 6's in 1/6 repeat forever. We can add these decimal numbers using Tiffany's strategy:

$$\begin{array}{r} 0.3333\ldots \\ +0.1666\ldots \\ \hline 0.4999\ldots \end{array}$$

So Tiffany's reasoning applied to 1/3 and 1/6 tells us that

$$\frac{1}{3} + \frac{1}{6} = .4999\ldots$$

where the 9's repeat forever.

On the other hand, we have

$$\begin{aligned}\frac{1}{3}+\frac{1}{6} &= \frac{2}{6}+\frac{1}{6}\\ &= \frac{3}{6}\\ &= \frac{1}{2}.\end{aligned}$$

This tells us that 1/3 + 1/6 = 1/2 = 0.5 does have a terminating decimal in addition to the repeating one found above.